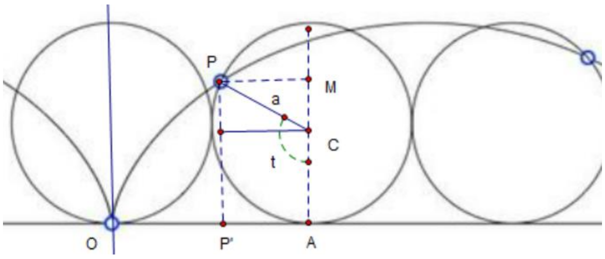
§ 擺線 the cycloid

圓心 C 的圓在直線上滾動，與直線的接觸點由 O 跑到 A，滾到 P，∠ACP=t



$\therefore \overline{OA} = AP = at$

$\overline{PM} = a\sin(\pi - t) = a\sin t$

P 點的 x 座標==$\overline{OA} - \overline{OP'} = \overline{OA} - \overline{PM} = a(t - \sin t)$

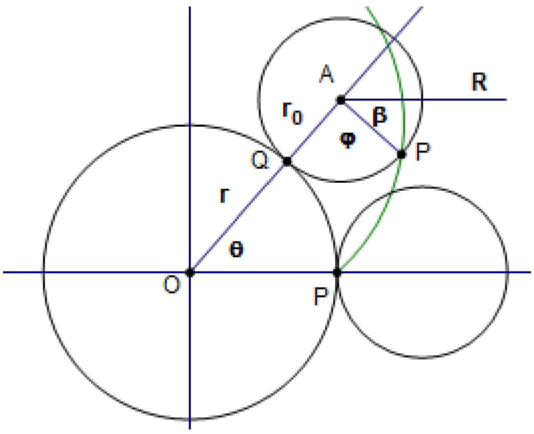
y 座標= $\overline{AC} + \overline{CM} = a + a\cos(\pi - t) = a(1 - \cos t)$

$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$$

$\int_0^{2\pi} \sqrt{(\dot{x})^2 + (\dot{y})^2}\, dt = 8a$ ；這裡有一個小小的習作 $\int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt = 8$

1. hypocycloid Rolling Hypocycloids [Azimuth]
2. 圓的藝術
3. Apollonan gasket
4. Enjoy Yourself Symbols
5. Sacred geometry---search in Pintreat

§ 外擺線(epicycloids)



$PQ = r_0\theta = r\varphi$

$\beta = \angle PAR = \pi - \theta - \varphi = \pi - (1 + \frac{r_0}{r})\theta$

$X(\theta) = \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

=

$[(r_0 + r)\cos\theta, (r_0 + r)\sin\theta] + [r\cos(-\beta), r\sin(-\beta)]$

假設取 $r_0 = 3, r = 1$,則 $\beta = \pi - 4\theta$,求弧長 $\int_0^{\frac{2\pi}{3}} \left|\frac{dX}{d\theta}\right| d\theta =$

$X(\theta) = [4\cos\theta - \cos 4\theta, 4\sin\theta - \sin 4\theta]$

$$\frac{dX}{d\theta}=[-4\sin\theta+4\sin 4\theta, 4\cos\theta-4\cos 4\theta]$$

$$\left|\frac{dX}{d\theta}\right|=4\sqrt{2-2\cos 3\theta}=4\sqrt{2}\sqrt{1-\cos 3\theta}$$

$\int_0^{\frac{2\pi}{3}}\sqrt{1-\cos 3\theta}\,d\theta=\frac{1}{3}\int_0^{2\pi}\sqrt{1-\cos\varphi}\,d\varphi$,令 $\varphi=3\theta$,則 $d\varphi=3d\theta$

$=\frac{\sqrt{2}}{3}\int_0^{2\pi}\left|\sin\frac{\varphi}{2}\right|d\varphi=\ldots=\frac{4\sqrt{2}}{3}$,所以 $\int_0^{\frac{2\pi}{3}}\left|\frac{dX}{d\theta}\right|d\theta=\frac{32}{3}$

[習作]算一下 $y=x^2$, $0\le x\le 1$ 的弧長

$$\int_0^1\sqrt{1+4x^2}\,dx=$$

查積分表 $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|+c$

$$\int_0^1\sqrt{1+4x^2}\,dx=\frac{1}{2}\int_0^2\sqrt{1+y^2}\,dy=\frac{1}{2}[\frac{y}{2}\sqrt{y^2+1}+\frac{1}{2}\ln\left|y+\sqrt{y^2+1}\right|_0^2]$$

$$=\frac{\sqrt{5}}{2}+\frac{1}{4}\ln(2+\sqrt{5})$$