弧長

(曲面上)曲線座標

$dx = X_u du + X_v dv$

則弧長 $ds^2 = dx \cdot dx = Edu^2 + 2Fdudv + Gdv^2$ 這是第一基本式 I

寫成 $ds^2 = dX \cdot dX = g_{ij} du^i du^j$

面積 $|X_u \wedge X_v| = \sqrt{EG - F^2}$

(兩向量 $\vec{u}, \vec{v}$ 所張的面積= $|\vec{u} \wedge \vec{v}| = \sqrt{|\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2}$)

例

C 是球面上的曲線 $X = [\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi]$

C: $\theta = \ln(\cot\frac{\varphi}{2}), 0 \le \varphi \le \frac{\pi}{2}$, 求 C 的長度

$X_\theta = [-\sin\theta \sin\varphi, \cos\theta \sin\varphi, 0]$

$X\varphi = [\cos\theta \cos\varphi, \sin\theta \sin\varphi, -\sin\varphi]$

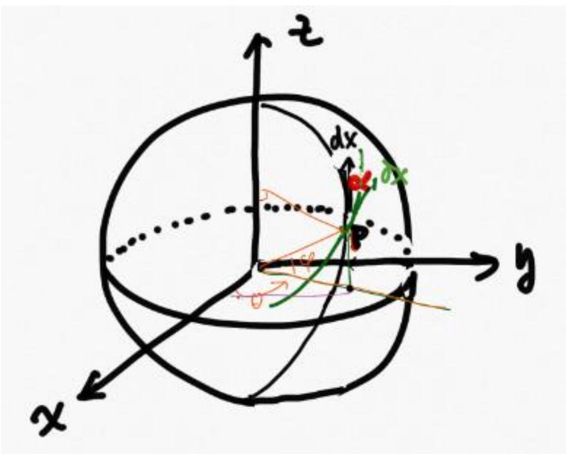
則 $E = X_\theta \cdot X_\theta = \sin^2\varphi$

F= $X_\theta \cdot X_\varphi = 0$, G= $X_\varphi \cdot X_\varphi = 1$

曲線長= $\int_0^{\frac{\pi}{2}} \sqrt{\sin^2\varphi \left(\frac{d\theta}{d\varphi}\right)^2 + 1} d\varphi = \int_0^{\frac{\pi}{2}} \sqrt{2} d\varphi = \frac{\pi}{\sqrt{2}}$

$\frac{d}{dx}\cot x = -\frac{1}{\sin^2 x}$

$\frac{d\theta}{d\varphi} = \frac{1}{\cot\frac{\varphi}{2}} (\frac{1}{2}) \cdot \frac{-1}{\sin^2\frac{\varphi}{2}} = \frac{-1}{\sin\varphi}$



例

球上面的斜航線(loxodrome)和經線(meridian)夾角= α (常數)

由赤道繞到北極的長度=

$$X=[a\cos\theta\cos\varphi, a\sin\theta\cos\varphi, a\sin\varphi]$$
$$X_\theta=[-a\sin\theta\cos\varphi, a\cos\theta\cos\varphi, 0]$$
$$X_\varphi=[-a\cos\theta\sin\varphi, -a\sin\theta\sin\varphi, a\cos\varphi]$$

$E=X_\theta\cdot X_\theta=a^2\cos^2\varphi$，F=0，G=$a^2$

經線　θ=常數　$dX=X_\theta d\theta+X_\varphi d\varphi=X_\varphi d\varphi$，$ds^2=a^2d\varphi^2$

斜航線　$\delta X=X_\theta\delta\theta+X_\varphi\delta\varphi$，$\delta s^2=a^2\cos^2\varphi\delta^2\theta+a^2\delta^2\varphi$

$$\cos\alpha=\frac{dx\cdot\delta x}{|dx||\delta x|}=\frac{a^2d\varphi\delta\varphi}{ad\varphi\cdot\sqrt{a^2\cos^2\varphi\delta^2\theta+a^2\delta^2\varphi}}$$
$$\cos^2\alpha(\cos^2\varphi\delta^2\theta+\delta^2\varphi)=\delta^2\varphi$$
$$\frac{\delta\varphi}{\delta\theta}=\cos\varphi\cot\alpha$$

斜航線的長度=$a\int_0^{\frac{\pi}{2}}\sqrt{\cos^2\varphi\delta^2\theta+\delta^2\varphi}=a\int_0^{\frac{\pi}{2}}\sqrt{\cos^2\varphi(\frac{\delta\theta}{\delta\varphi})^2+1}\delta\varphi$

$$=a\int_0^{\frac{\pi}{2}}\sec\alpha\delta\varphi=\frac{a\pi\sec\alpha}{2}$$

取 $X(\theta,\varphi)=[\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta]$時
$$\frac{\delta\varphi}{\delta\theta}=\frac{\tan\beta}{\sin\theta}$$
$$\int\frac{1}{\sin x}dx=-\ln|\csc x+\cot x|+c=-\ln\left|\cot\frac{x}{2}\right|+c$$
$$\varphi=-\tan\beta\ln\left|\cot\frac{\theta}{2}\right|+c$$
$$-(\varphi+c)\cot\beta=\ln\left|\cot\frac{\theta}{2}\right|=-\ln\left|\tan\frac{\theta}{2}\right|$$
$$\ln(\tan\frac{\theta}{2})=\pm(\theta+c)\cot\beta$$
這是斜航線(loxodrome)的方程式