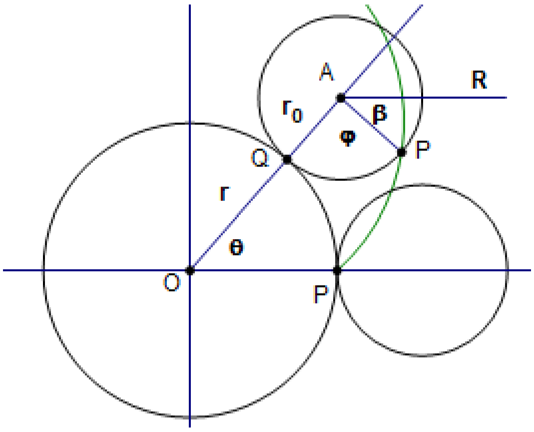
§01 弧長 X=(x(u),y(u),z(u))

則 s(t) = $\int_{u_0}^{u}\sqrt{\dot{X}\cdot\dot{X}}\,du$

平面上 $ds=\sqrt{dx^2+dy^2}=\sqrt{1+(\frac{dy}{dx})^2}\,dx$

曲面上 $dX=X_u du+X_v dv$，$ds^2=dX\cdot dX=g_{ij}dx^i dx^j$

例 外擺線(epicycloids)



$\widehat{PQ}=r_0\theta=r\varphi$

$\beta=\angle PAR=\pi-\theta-\varphi=\pi-(1+\frac{r_0}{r})\theta$

$X(\theta)=\overline{OP}=\overline{OA}+\overline{AP}$

$=[(r_0+r)\cos\theta,(r_0+r)\sin\theta]+[r\cos(-\beta),r\sin(-\beta)]$

假設取 $r_0=3, r=1$，則 $\beta=\pi-4\theta$，求弧長 $\int_0^{\frac{2\pi}{3}}\left|\frac{dX}{d\theta}\right|d\theta=$

$X(\theta)=[4\cos\theta-\cos 4\theta, 4\sin\theta-\sin 4\theta]$

$\frac{dX}{d\theta}=[-4\sin\theta+4\sin 4\theta, 4\cos\theta-4\cos 4\theta]$

$\left|\frac{dX}{d\theta}\right|=4\sqrt{2-2\cos 3\theta}=4\sqrt{2}\sqrt{1-\cos 3\theta}$

$\int_0^{\frac{2\pi}{3}}\sqrt{1-\cos 3\theta}\,d\theta=\frac{1}{3}\int_0^{2\pi}\sqrt{1-\cos\varphi}\,d\varphi$，令 $\varphi=3\theta$，則 $d\varphi=3d\theta$

$=\frac{\sqrt{2}}{3}\int_0^{2\pi}\left|\sin\frac{\varphi}{2}\right|d\varphi=...=\frac{4\sqrt{2}}{3}$，所以 $\int_0^{\frac{2\pi}{3}}\left|\frac{dX}{d\theta}\right|d\theta=\frac{32}{3}$

[習作]算一下 $y=x^2$，$0\le x\le 1$的弧長

$\int_0^1\sqrt{1+4x^2}\,dx=$

查積分表 $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right|+c$

$\int_0^1\sqrt{1+4x^2}\,dx=\frac{1}{2}\int_0^2\sqrt{1+y^2}\,dy=\frac{1}{2}[\frac{y}{2}\sqrt{y^2+1}+\frac{1}{2}\ln\left|y+\sqrt{y^2+1}\right|_0^2]$

$=\frac{\sqrt{5}}{2}+\frac{1}{4}\ln(2+\sqrt{5})$