高斯絕妙定理是高斯在 1827 年證明的。
此定理說高斯曲率是曲面的內秉(intrinsic)性質，由第一基本式決定。
第一基本式 $ds^2 = Edu^2 + 2Fdudv + Gdv^2$

在曲面 M 上，若有局部坐標系(u，v)其所有 u 曲線都是 M 上的么速測地線，且 u 曲線與 v 曲線在相交的地方都互相垂直，則稱(u，v)是 M 上的一個(局部)測地坐標系。
即 $ds^2 = du^2 + Gdv^2$　初等微分幾何講稿 p.116 這裡有測地坐標系的製作；
大域微分幾何(上) p.15

§ 定理(Gauss Theorema Egregium)

Gauss 曲率 $K = -\dfrac{g_{uu}}{g}$

(或者寫成 $K = -\dfrac{1}{\sqrt{G}}\dfrac{\partial^2\sqrt{G}}{\partial u^2}$ ，其中 $g = \sqrt{G}$)

證明

建立測地坐標系， $g^2 = G$

$\dfrac{\partial}{\partial u}X_{uv} = \dfrac{\partial}{\partial v}X_{uu}$ ，兩邊對 X_v 做內積，故事從這裡說起。

$< \dfrac{\partial}{\partial u}X_{uv}, X_v > = < \dfrac{\partial}{\partial v}X_{uu}, X_v >$ ...(*)

$\because \dfrac{\partial}{\partial u} < X_{uv}, X_v > = < \dfrac{\partial}{\partial u}X_{uv}, X_v > + < X_{uv}, X_{uv} >$

$\therefore$ (*)的左式 $< \dfrac{\partial}{\partial u}X_{uv}, X_v > = \dfrac{\partial}{\partial u} < X_{uv}, X_v > - < X_{uv}, X_{uv} >$

同理，右式 $< \dfrac{\partial}{\partial v}X_{uu}, X_v > = \dfrac{\partial}{\partial v} < X_{uu}, X_v > - < X_{uu}, X_{vv} >$

$g^2 = X_v \cdot X_v$ ， $\dfrac{\partial}{\partial u}g^2 = 2X_{uv}\cdot X_v$ ， $\therefore X_{uv}\cdot X_v = \dfrac{1}{2}\dfrac{\partial}{\partial u}g^2 = gg_u$

$\dfrac{\partial}{\partial u} < X_{uv}, X_v > = \dfrac{\partial}{\partial u}gg_u = {g_u}^2 + gg_{uu}$

此時建立的 orthonormal frame 是 $X_1 = X_u, X_2 = \dfrac{X_v}{g}, X_3 = N = X_1 \times X_2$

X_{uv} 表成 X_u, X_v, N 的線性組合

$X_{uv} = < X_{uv}, X_u > X_u + < X_{uv}, X_v > \dfrac{X_v}{g^2} + < X_{uv}, N > N$

$\because X_u \cdot X_u = 1, \therefore X_{uv}\cdot X_u = 0$

$X_{uv}\cdot X_{uv}=<X_{uv},X_v>^2\dfrac{X_v\cdot X_v}{g^4}+<X_{uv},N>^2=<X_{uv},X_v>^2\dfrac{1}{g^2}+<X_{uv},N>^2$

$=(gg_u)^2\times\dfrac{1}{g^2}+<X_{uv},N>^2=g_u{}^2+\bar{f}^{\,2}$

$\because X_{uu}$ 只有法部，$\therefore X_{uu}=<X_{uu},N>N$

$<X_{uu},X_{vv}>=<<X_{uu},N>N,X_{vv}>=<X_{uu},N><X_{vv},N>=\bar{e}\,\bar{g}$

(此處 $\bar{g}=X_{vv}\cdot N$ ，$K=\dfrac{\bar{e}\,\bar{g}-\bar{f}^2}{EG-F^2}$ 。)

所以(*)的左式= $g_u{}^2+gg_{uu}-g_u{}^2-\bar{f}^2=gg_{uu}-\bar{f}^{\,2}$

右式= $\dfrac{\partial}{\partial v}<X_{uu},X_v>-<X_{uu},X_{vv}>$ ，$X_{uu}=<X_{uu},N>N$

$=-<<X_{uu},N>N,X_{vv}>=-<X_{uu},N><X_{vv},N>=-\bar{e}\,\bar{g}$

$gg_{uu}-\bar{f}^{\,2}=-\bar{e}\,\bar{g}$ ，$g^2=G$

$g_{uu}=-\dfrac{\bar{e}\,\bar{g}-\bar{f}^2}{\sqrt{G}}$ ，所以 $K=\dfrac{\bar{e}\,\bar{g}-\bar{f}^2}{EG-F^2}=\dfrac{\bar{e}\,\bar{g}-\bar{f}^{\,2}}{G}=-\dfrac{g_{uu}}{g}$

以上即 semigeodesic coordinates $g_{11}=1,g_{12}=0,g_{22}=G$ p.183

此時 $\Gamma_{11}^1=\Gamma_{11}^2=\Gamma_{12}^1=0$ ，$\Gamma_{12}^2=\dfrac{1}{2G}\dfrac{\partial G}{\partial u}$ ，$\Gamma_{22}^1=-\dfrac{1}{2}\dfrac{\partial G}{\partial u},\Gamma_{22}^2=\dfrac{1}{2G}\dfrac{\partial G}{\partial v}$

Where $\Gamma_{ij}^k=\dfrac{1}{2}g^{kl}(\dfrac{\partial g_{jl}}{\partial u^i}+\dfrac{\partial g_{il}}{\partial u^j}-\dfrac{\partial g_{ij}}{\partial u^l})$

Abraham Goetz 的書是用 Christoffel symbol 處理。 p.186

§ Introduction to Dfferential Geometry Abraham Goetz p.188

For isothermic coordinates，$g_{11}=g_{22}=\rho^2,g_{12}=0$

$\Gamma_{11}^1=\Gamma_{12}^2=-\Gamma_{22}^1=\dfrac{\partial\ln\rho}{\partial u}$

$\Gamma_{12}^1=-\Gamma_{22}^1=-\Gamma_{11}^2=\dfrac{\partial\ln\rho}{\partial v}$

$K=-\dfrac{1}{\rho^2}\Delta\rho$ ，where $\Delta f=\dfrac{\partial^2 f}{\partial u^2}+\dfrac{\partial^2 f}{\partial v^2}$ is the Laplace operator

§ 建立一個測地坐標系，稍懂。

那麼，如何用 exponential map 建構一個 Riemann normal coordinates(簡正座標) at p。

Now we see for any $p \in M$, there exists a neighborhood $U \subset M$ of p and a neighborhood $V \subset T_pM$ of 0 so that the exponential map

$$\exp_p : V \to U$$

is a diffeomorphism. But T_pM, and thus V, is Euclidian, so the triple $\{\exp_p^{-1}, U, V\}$ form a local chart of M near p. Usually one takes such U's to be geodesic balls (and thus V to be Euclidean balls). We will fix an orthonormal basis $\{e_i\}$ of T_pM, which gives us linear coordinates for V. We denote the corresponding coordinate functions on U by $\{x^i\}$.

Definition 1.1. The local chart $\{U; x^1, \cdots, x^m\}$ described above is called *normal coordinate system* at p.

Lemma

Let $\{\mathrm{U}; x^1, x^2, \ldots, x^m\}$ be a normal coordinate system at p。Then for all $1 \le i, j, k \le m$

(1) $g_{ij}(p) = \delta_{ij}$ (2) $\Gamma_{ij}^k = 0$ (3) $\partial_k g_{ij}(p) = 0$