§ 共軛極小曲面

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Differential Geometry of Curves and Surfaces p.213

When two differentiable functions $f,g:U\subset R^2\to R$ satisfy the Cauchy Riemann equations $\frac{\partial f}{\partial u}=\frac{\partial g}{\partial v},\frac{\partial f}{\partial v}=-\frac{\partial g}{\partial u}$,

They are easily seen to be harmonic；in this situation，f and g are said to be harmonic conjugate。

Let x and y be isothermal parametrizations of minimal surfaces such that their component functions are pairwise harmonic conjugate；then x and y are called conjugate minimal surfaces。Prove that

1. The helicoid and the catenoid are congugate minimal surfaces
2. Given two conjugate minimal surfaces family，x and y，the surface $z=(\cos t)x+(\sin t)y$ $\cdots$(*)is again minimal for all $t\in R$
3. All surfaces of one-parameter family (*) have the same fundamental form：$E=<x_u,x_u>=<y_v,y_v>,F=0,G=<x_v,x_v>=<y_u,y_u>$

Thus，any two conjugate minimal surfaces can be joined through a one-paraeter family of minimal surfaces，and the first fundamental of this family is independent of t。

X=X(u,v) is said to be isothermal(等溫)，if $<X_u,X_u>=<X_v,X_v>$ and $<X_u,X_v>=0$ p.201

1. X(u,v)=(acoshvcosu,acoshvsinu,av)
   Y(u,v)=(asinhvcosu,asinhvsinu,au)

Let $f(u,v)=a\cosh v\sin u, g(u,v)=a\sinh v\cos u$

Then $\frac{\partial f}{\partial u}=a\cosh v\cos u=\frac{\partial g}{\partial v}$ ，$\frac{\partial f}{\partial v}=a\sinh v\sin u=-\frac{\partial g}{\partial u}$

2． $Z=(\cos t)X+(\sin t)Y$ ，則

$Z_u=(\cos t)X_u+(\sin t)Y_u$

$Z_v=(\cos t)X_v+(\sin t)Y_v$

$$\langle Z_u,Z_u\rangle=(\cos^2 t)\langle X_u,X_u\rangle+(\sin^2 t)\langle Y_u,Y_u\rangle+2\sin t\cos t\langle X_u,Y_u\rangle$$

$$<Z_v,Z_v>=\cos^2 t<X_v,X_v>+\sin^2 t<Y_v,Y_v>+2\sin t\cos t<X_v,Y_v>$$

其中$\langle X_u,Y_u\rangle=\langle Y_v,Y_u\rangle=0$，前者是 harmonic conjugate 後者是 isothermal 的條件，又 同理$\langle X_u,X_u\rangle=\langle Y_v,Y_v\rangle$ ，所以

$$\langle Z_u, Z_u \rangle = (\cos^2 t)\langle X_u, X_u \rangle + (\sin^2 t)\langle Y_u, Y_u \rangle = (\cos^2 t)\langle Y_v, Y_v \rangle + (\sin^2 t)\langle Y_u, Y_u \rangle$$
$$= (\cos^2 t + \sin^2 t)\langle Y_u, Y_u \rangle = \langle Y_u, Y_u \rangle$$

Independent of t

It is easy to check that $Z_{uu} + Z_{vv} = 0$

The catenoid is locally isometric to the helicoid。 P.221

懸鏈曲面可以保長地連續變換到螺旋曲面，過程中每一點的高斯曲率不變。