高斯 Carl Friedrich Gauss 1777-1855

$$\iint_M KdA+\int_{\partial M} k_g ds = 2\pi\chi(M)$$

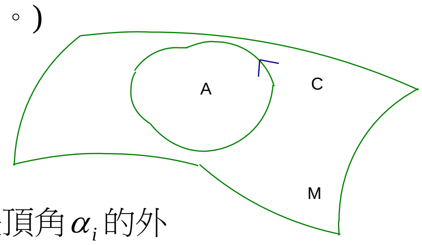
高斯-博內定理與曲面的拓撲性質有關，
它的特例(在平面上)稱為切線轉角定理。
陳省身先生在 1944 年把它推廣了，在物理上有好幾個應用。M is a closed oriented Riemannian manifold with an even dimension d，then $\int_M \Omega = \chi(M)$ where $\Omega_i^j = R_{ikj}^j d\omega^k \wedge d\omega^l$

[Oliver Knill](https://www.youtube.com/watch?v=-UNkiKDidI8)先生[圖論的 Gauss-Nonnet-Chern 定理]
https://www.youtube.com/watch?v=-UNkiKDidI8
Something interesting！

§01 敘述

1. C 是曲面 M 上的 Jordan curve(平滑、封閉、簡單。)

   則 $\iint_A KdA + \int_C \kappa_g ds = \int_C d\theta = 2\pi$



2. 若 C 是分段平滑的,A compact,oriented,則

   $\iint_A KdA + \int_C \kappa_g ds + \sum_i (\pi - \alpha_i) = 2\pi\aleph$,其中 $\pi - \alpha_i$ 是頂角 α_i 的外角，$\aleph$是 Euler 示性數。

§02 預備定理

1. 何謂(1)測地曲率 geodesic curvature (2)高斯曲率

   $$\frac{dT}{ds} = \kappa_n N + \kappa_g U \text{ ,} U = N \times T$$

   則 $\kappa_g = \frac{dT}{ds} \cdot U$ 是為測地曲率

   半徑為 r 的球面 $K = \frac{1}{r^2}$

2. Green 定理

   C 是 piecewise smooth Jordan curve

   則 $\iint_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy = \oint_C Pdx + Qdy$

   例 $C: x^2 + \frac{y^1}{4} = 1$,力場 F(x,y)=[3x+y,-x+2y]加諸質點 P,沿 C 逆時針繞一圈所作的功 $W = \oint_C F \cdot dx = \oint Pdx + Qdy = ... = -4\pi$

3. 取 lines of curvature 作參數曲線時

$$K=\frac{eg-f^2}{EG-F^2}=\frac{1}{-\sqrt{EG}}\left\{\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)\right\}$$

4. Liouville's lemma

u-curve，v-curve 在曲面 M 上形成一正交網，$T_P(M)$由 T,U 所張。

假設 T 與 u-curve 夾角為θ ，則$\begin{cases} T=i_1\cos\theta+i_2\sin\theta \\ U=-i_1\sin\theta+i_2\cos\theta \end{cases}, T\cdot i_1=\cos\theta$

$$ds_1=ds\cos\theta, ds_2=ds\sin\theta, ds^2=ds_1^{\ 2}+ds_2^{\ 2}$$

定義$\kappa_g=\frac{dT}{ds}\cdot U$

則$\kappa_g=\kappa_1\cos\theta+\kappa_2\sin\theta+\frac{d\theta}{ds}$,其中$\kappa_1=(\kappa_g)_{u-curve}=\frac{di_1}{ds_1}\cdot i_2=-i_1\cdot\frac{di_2}{ds_1}$

$$\kappa_2=(\kappa_g)_{v-curve}=\frac{-di_2}{ds_2}\cdot i_1=i_2\cdot\frac{di_1}{ds_2}$$

$$\frac{di_1}{ds}=\frac{di_1}{ds_1}\frac{ds_1}{ds}+\frac{di_1}{ds_2}\frac{ds_2}{ds}=\cos\theta\frac{di_1}{ds_1}+\sin\theta\frac{di_1}{ds_2}$$

$$\frac{di_2}{ds}=...=\cos\theta\frac{di_2}{ds_1}+\sin\theta\frac{di_2}{ds_2}$$

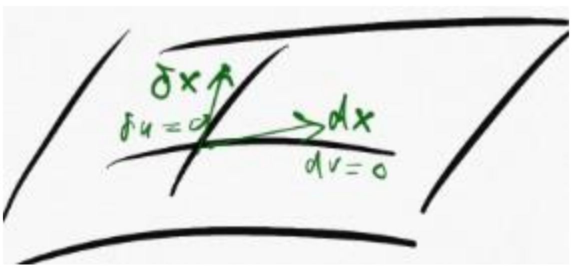
$$U=-i_1\sin\theta+i_2\cos\theta$$

$$\frac{di_1}{ds}\cdot U=\kappa_1\cos^2\theta+\kappa_2\sin\theta\cos\theta, \frac{di_2}{ds}\cdot U=\kappa_1\sin\theta\cos\theta+\kappa_2\sin^2\theta$$

$$\frac{dT}{ds}=\cos\theta\frac{di_1}{ds}+\sin\theta\frac{di_2}{ds}+U\frac{d\theta}{ds}$$

$$\frac{dT}{ds}\cdot U=...=\kappa_1\cos\theta+\kappa_2\sin\theta+\frac{d\theta}{ds}$$

§03 證明



選一正交坐標系 F=0

$$dA=|dx\times\delta x|=\sqrt{EG-F^2}du\delta v=\sqrt{EG}du\delta v$$

在 u-curve 上,

$$ds_1^{\ 2}=Edu^2+2Fdudv+Gdv^2=Edu^2$$

在 v-curve 上,

$$ds_2^{\ 2}=...=Gdv^2, ds_2=\sqrt{G}dv$$

所以 $ds_1 = \sqrt{E}du = ds\cos\theta, ds_2 = \sqrt{G}dv = ds\sin\theta$,由 Liouville's lemma

$$\kappa_g ds = d\theta + \kappa_1\cos\theta ds + \kappa_2\sin\theta ds = d\theta + \kappa_1\sqrt{E}du + \kappa_2\sqrt{G}dv$$

在 u-curve 上, $\frac{dv}{ds} = 0, \frac{du}{ds} = \frac{1}{\sqrt{E}}, \kappa_1 = \Gamma_{11}^2\frac{\sqrt{EG-F^2}}{E\sqrt{E}}, (\Gamma_{11}^2 = \frac{-E_v}{2G}) = -\frac{E_v}{2E\sqrt{G}}$

在 v-curve 上,du=0, $\frac{dv}{ds} = \frac{1}{\sqrt{G}}, \kappa_2 = ... = \frac{G_u}{2G\sqrt{E}}$

$$\int_C \kappa_1\sqrt{E}du + \kappa_2\sqrt{G}dv = \iint_A [\frac{\partial}{\partial u}(\kappa_2\sqrt{G}) - \frac{\partial}{\partial v}(\kappa_1\sqrt{E})]dudv \ldots\ldots(※)$$

$$= \iint_A [\frac{\partial}{\partial u}(\frac{G_u}{2\sqrt{EG}}) + \frac{\partial}{\partial v}(\frac{E_v}{2\sqrt{EG}})]dudv$$

$$= \iint_A -K\sqrt{EG}dudv = -\iint_A KdA$$

(F=0 時, $K = -\frac{1}{\sqrt{EG}}\left\{\frac{\partial}{\partial u}(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}) + \frac{\partial}{\partial v}(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v})\right\}$ ……尚待證明

$$= -\frac{1}{\sqrt{EG}}\left\{\frac{\partial}{\partial u}(\frac{G_u}{2\sqrt{EG}}) + \frac{\partial}{\partial v}(\frac{E_v}{2\sqrt{EG}})\right\})$$

所以 當 C 是 M 上一 Jardon Curve

$$\int_C \kappa_g ds = \int_C d\theta - \iint_A KdA$$

$$\iint_A KdA + \int_C \kappa_g ds = \int_C d\theta = 2\pi$$

取一平行向量場 $Z = \cos\alpha e_1 + \sin\alpha e_2$ ， $(\frac{dZ}{dt})^T = 0$ ，證明 $\frac{d\alpha}{dt} = -g_u\frac{dv}{dt}$

$$\frac{dZ}{dt} = (-\sin\alpha e_1 + \cos\alpha e_2)\frac{d\alpha}{dt} + (\cos\alpha\frac{de_1}{dt} + \sin\alpha\frac{de_2}{dt})$$

$0 = \frac{dZ}{dt}\cdot e_2 = \cos\alpha\frac{d\alpha}{dt} + \cos\alpha(\frac{de_1}{dt}\cdot e_2)$ ，所以

$$\frac{d\alpha}{dt} = -\frac{de_1}{dt}\cdot e_2 = -(\frac{\partial X_u}{\partial u}\frac{du}{dt} + \frac{\partial X_u}{\partial v}\frac{dv}{dt})\cdot\frac{X_v}{g}$$

$$= -\langle X_{uu}, \frac{X_v}{g}\rangle\frac{du}{dt} - \langle X_{uv}, \frac{X_v}{g}\rangle\frac{dv}{dt}$$

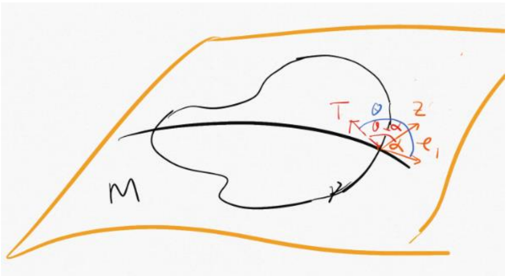
$$= 0 - X_{uv}\cdot X_v\frac{1}{g}\frac{dv}{dt} \quad (X_{uv}\cdot X_v = \frac{\partial}{\partial u}(\frac{1}{2}g^2))$$

$$= -g_u \frac{dv}{dt}$$

Green 定理 $\int_\gamma Pdu + Qdv = \iint_A (\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v})dudv$

$$\delta_\gamma \alpha = \int_a^b (\frac{d\alpha}{dt})dt = \int_a^b (-g_u \frac{dv}{dt})dt = \int_\gamma -g_u dv = \iint_A -g_{uu} dudv$$

$$= \iint_A \frac{-g_{uu}}{g} d\sigma = \iint_A K d\sigma \text{ (因為 } d\sigma = gdudv \text{)}$$



$T = \frac{d\lambda}{ds} = \cos\theta e_1 + \sin\theta e_2$

由切線轉角定理 $\delta_\gamma \theta = 2\pi$

沿 γ 的平行向量場 $Z = \cos\alpha e_1 + \sin\alpha e_2$

$\theta = \alpha + (\theta - \alpha)$

$$2\pi = \delta_\gamma \theta = \delta_\gamma \alpha + \delta_\gamma (\theta - \alpha) = \iint_A K d\sigma + \int_\gamma \frac{d}{ds}(\theta - \alpha) ds$$

以下證明 $\frac{d}{ds}(\theta - \alpha) = \kappa_g$

$T = \cos\theta e_1 + \sin\theta e_2, N = -\sin\theta e_1 + \cos\theta e_2$

$$\frac{dT}{ds} = (-\sin\theta e_1 + \cos\theta e_2)\frac{d\theta}{ds} + \cos\theta \frac{de_1}{ds} + \sin\theta \frac{de_2}{ds}$$

$$\kappa_g = (\frac{dT}{ds})^T = \frac{dT}{ds} \cdot N$$

$$= (\sin^2\theta + \cos^2\theta)\frac{d\theta}{ds} + (-\sin^2\theta \frac{de_2}{ds} \cdot e_1 + \cos^2\theta \frac{de_1}{ds} \cdot e_2)$$

$$= \frac{d\theta}{ds} + \frac{de_1}{ds} \cdot e_2 = \frac{d\theta}{ds} - \frac{d\alpha}{ds} = \frac{d}{ds}(\theta - \alpha)$$

得 $\iint_A K d\sigma + \int_\gamma \kappa_g ds = 2\pi$

曲面 M:X=X[u,v]，以下建立么速測地坐標系，

$X_u = \frac{\partial X}{\partial u}, X_v = \frac{\partial X}{\partial v}$ ，使得 E=1，F=0， $ds^2 = du^2 + Gdv^2$

取 $e_1 = X_u, e_2 = X_v / g, e_3 = \frac{X_u \times X_v}{g} = N$ ， $g^2 = G = X_v \cdot X_v$

$(|e_2| = \left|\frac{X_v \cdot X_v}{g^2}\right| = \frac{G}{g^2} = 1)$

$$K = \kappa_1\kappa_2 = \frac{N_u \times N_v}{X_u \times X_v} = \frac{\det(b_{ij})}{\det(g_{ij})} = \frac{eg - f^2}{EG - F^2}$$

$$N_u =< N_u, e_1 > e_1 + < N_u, e_2 > e_2$$

$$N_v =< N_v, e_1 > e_1 + < N_v, e_2 > e_2$$

$$N_u \times N_v = \{< N_u, e_1 >< N_v, e_2 > - < N_u, e_2 >< N_v, e_1 >\} e_1 \times e_2$$

$$= \frac{1}{g^2}\{< N_u, X_u >< N_v, X_v > - < N_u, X_v >< N_v, X_u >\} X_u \times X_v$$

$$= \frac{1}{g^2}\{< N, X_{uu} >< N, X_{vv} > - < N, X_{uv} >^2\} X_u \times X_v$$

(由 $N \cdot X_u = 0$ ，得 $N_u \cdot X_u = -N \cdot X_{uu}$ 。由 $N \cdot X_v = 0$ ，得 $N_v \cdot X_v = -N \cdot X_{vv}$

同理 $N_u \cdot X_v = N_v \cdot X_u = -N \cdot X_{uv}$)

由 $\frac{\partial}{\partial u} X_{uv} - \frac{\partial}{\partial v} X_{uu} = 0$ ，兩邊對 X_v 作內積

$$< \frac{\partial}{\partial u} X_{uv}, X_v > - < \frac{\partial}{\partial v} X_{uu}, X_v >= 0$$

$$(\frac{\partial}{\partial u} < X_{uv}, X_v >=< \frac{\partial}{\partial u} X_{uv}, X_v > + < X_{uv}, X_{uv} >)$$

$$\frac{\partial}{\partial u} < X_{uv}, X_v > - < X_{uv}, X_{uv} > - \frac{\partial}{\partial v}(X_{uu}, X_v) + < X_{uu}, X_{vv} >= 0 \;\ldots\ldots(※)$$

$$g^2 = X_v \cdot X_v$$

$\frac{\partial}{\partial u} g^2 = X_{uv} \cdot X_v + X_v \cdot X_{uv}$ ，所以 $X_{uv} \cdot X_v = \frac{1}{2}\frac{\partial}{\partial u} g^2 = gg_u$

$$\frac{\partial}{\partial u} < X_{uv}, X_v >= \frac{\partial}{\partial u} gg_u = g_u{}^2 + gg_{uu} \ldots(1)$$

$$X_{uv} =< X_{uv}, X_u > X_u + < X_{uv}, X_v > \frac{X_v}{g^2} + < X_{uv}, N > N$$

$$- < X_{uv}, X_{uv} >= -\frac{1}{g^2} < X_{uv}, X_v >^2 - < X_{uv}, N >^2 \ldots(2)$$

(因為 $g^2 = X_v \cdot X_v, \dfrac{\partial}{\partial u} g^2 = X_{uv} \cdot X_v + X_v \cdot X_{uv} = 2X_{uv} \cdot X_v$)

$= -\dfrac{1}{g^2}(\dfrac{\partial}{\partial u}(\dfrac{1}{2}g^2))^2 - <X_{uv}, N>^2 = -g_u{}^2 - <X_{uv}, N>^2$

$\dfrac{\partial}{\partial v} <X_{uu}, X_v> = 0 \ldots (3)$

(因為 $X_{uu} = \_\_ X_u + \_\_ X_v + \_\_ N = \_\_ N$ ，$<X_v, N> = 0$)

X_{uu} 只有法部，所以 $X_{uu} = <X_{uu}, N> N$

$<X_{uu}, X_{vv}> = << X_{uu}, N> N, X_{vv}> = <X_{uu}, N><X_{vv}, N> \ldots (4)$

(※)變成

$gg_{uu} - <X_{uv}, N> + <X_{uu}, N><X_{vv}, N> = 0$

其中 $-<X_{uv}, N>^2 + <X_{uu}, N><X_{vv}, N> = g^2 K$

所以 $K = \dfrac{-g_{uu}}{g}$

§ 04 應用

1. Hairy ball theorem：There does not exist a non-vanishing continuous tangent vector field X on a sphere S^2
2. The Poincare-Hopf index theorem
   Let X be a differentiable tangent vector field on a compact surface S which has only finitely many isolated singular points $p_1, ..., p_n$ 。Then $\sum_i \mu(p_i) = \chi(S)$
   $\mu(p)$ is the index of singular point p。
3. Jacobi 定理
4. 一個正曲率的緊緻(compact)曲面與球面同胚
5. T是一個測地線三角形，則其三內角和= (1) $=\pi$ if K=0 (2) $>\pi$ if K>0 (3) $<\pi$ if K<0

§05 討論

1. 對一個平面上的曲線，K=0，回到切線轉角定理。
2. 半徑=a 的球面,平面 E 到球心的距離=d，和球面截出一圓 C，P 是 C 上一點，求 $\kappa_g(P) = \ldots\ldots \dfrac{d}{ar}, (r = \sqrt{a^2 - d^2})$ 是小圓半徑

$$X(\theta,\varphi)=[a\cos\theta\cos\varphi, a\cos\theta\sin\varphi, a\sin\theta]$$

$$X_\theta=[-a\sin\theta\cos\varphi, -a\sin\theta\sin\varphi, a\cos\theta]$$

$$X_\varphi=[-a\cos\theta\sin\varphi, a\cos\theta\cos\varphi, 0]$$

$$E=a^2, F=0, G=a^2\cos^2\theta$$

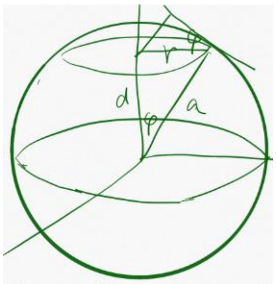
$$dA=\sqrt{EG-F^2}\,d\theta d\varphi=a^2\cos\theta d\theta d\varphi$$

$$\iint_A KdA=\int_0^{2\pi}\int_{\theta_0}^{\frac{\pi}{2}}\cos\theta d\theta d\varphi \text{ ,其中} \sin\theta_0=\frac{d}{a}$$

$$=2\pi(1-\frac{d}{a})$$

$$\int_C \kappa_g ds=2\pi r\kappa_g=2\pi-2\pi(1-\frac{d}{a})=2\pi\frac{d}{a} \text{ ,所以} \kappa_g=\frac{d}{ar}$$

another viewpoint



小圓的曲率= $\frac{1}{r}$ 在切平面的投影

$$\kappa_g=\frac{1}{r}\cos\varphi=\frac{1}{r}\times\frac{d}{a}=\frac{d}{ar}$$

Consider a polar cap S on a round sphere of radius 1，

Then Area S=

$$\text{Area } S = \int_0^\theta 2\pi\sin\theta\ d\theta = 2\pi(1-\cos\theta),$$

hence $\int_S K\ d(\text{area}) = 1 \times \text{Area } S = 2\pi(1-\cos\theta)$.

The curvature κ of ∂S = 1/radius = $1/\sin\theta$.

The geodesic curvature κ_g of ∂S is $\cos\theta/\sin\theta$, hence

$$\int_{\partial S}\kappa_g\ ds = \kappa_g \times \text{length}(\partial S)$$

$$= (\cos\theta/\sin\theta)\ 2\pi\sin\theta = 2\pi\cos\theta.$$

Therefore

$$\int_S K\ d(\text{area}) + \int_{\partial S}\kappa_g\ ds = 2\pi(1-\cos\theta) + 2\pi\cos\theta$$

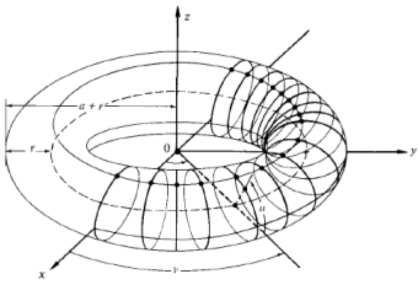
$$= 2\pi = 2\pi\chi(S).$$

§ 06 Euler 特徵數

另一個重要的不變量是尤拉特徵數：多面體的
V-E+F=2
在[笛卡爾的秘密手記]一書中，據稱他發現了尤拉特徵數，基於某種原因他並沒有發表。
笛卡爾有記錄清醒夢(lucid dream)的習慣，1637 年他因為清醒夢創立了坐標系。
這裡有一個研究清醒夢的平台。
透過感官引發清明夢的功夫稱太玄功，姑且聽之。

§ Gaussian Curvature of a Torus

Example 4.2. A torus T is created by rotating a radius r circle around a straight line distance a $(a > r)$ away. Consider a particular parametrization (Figure 5):
$\mathbf{x}(u,v) = ([a + rcos(u)]cos(v), [a + rcos(u)]sin(v), rsin(u))$ $where\ u, v \in (0, 2\pi)$



$$\mathbf{x}_u = (-rsin(u)cos(v), -rsin(u)sin(v), rcos(u))$$
$$\mathbf{x}_v = (-[a + rcos(u)]sin(v), [a + rcos(u)]cos(v), 0)$$
$$\mathbf{x}_{uu} = (-rcos(u)cos(v), -rcos(u)sin(v), -rsin(u))$$
$$\mathbf{x}_{uv} = (rsin(u)sin(v), -rsin(u)cos(v), 0)$$
$$\mathbf{x}_{vv} = (-[a + rcos(u)]cos(v), -[a + rcos(u)]sin(v), 0)$$

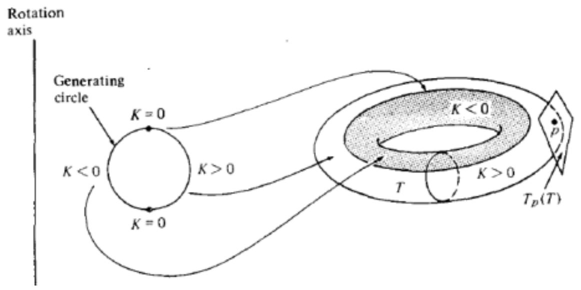
As a result, $E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = r^2,\ F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0,\ G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = [a + rcos(u)]^2.$

$$\begin{aligned} e &= -\langle N_u, \mathbf{x}_u \rangle = \langle N, \mathbf{x}_{uu} \rangle \\ &= \left\langle \frac{\mathbf{x}_v \wedge \mathbf{x}_u}{|\mathbf{x}_v \wedge \mathbf{x}_u|}, \mathbf{x}_{uu} \right\rangle \\ &= \frac{det(\mathbf{x}_u, \mathbf{x}_v, \mathbf{x}_{uv})}{\sqrt{EG - F^2}} \\ &= \frac{r^2[a + rcos(u)]}{r[a + rcos(u)]} \\ &= r \end{aligned}$$

Similarly, $f = 0, g = cos(u)[a + rcos(u)]$. Therefore,

$$K = \frac{eg - f^2}{EG - F^2} = \frac{cos(u)}{r[a + rcos(u)]}.$$

As Figure 6 below illustrates, the torus T can be divided into 3 parts by the signs of the Gaussian curvature:
1. $K > 0$ when $u \in \left(0, \frac{\pi}{2}\right)$ or $u \in \left(\frac{3}{2}\pi, 2\pi\right)$
2. $K = 0$ when $u = \frac{\pi}{2}$ or $u = \frac{3}{2}\pi$
3. $K < 0$ when $u \in \left(\frac{\pi}{2}, \frac{3}{2}\pi\right)$



Theorem 6.7 (The Global Gauss-Bonnet Theorem). .
Let $R \subset S$ be a regular region of an oriented surface and let $C_1, ..., C_n$ be the closed, simple, piecewise regular curves which form the boundary ∂R of R. Suppose that each C_i is positively oriented and let $\theta_1, ...\theta_p$ be the set of all external angles of the curves $C_1, ..., C_n$. Then

$$\sum_{i=1}^{n} \int_{C_i} k_g(s)ds + \iint_R K d\sigma + \sum_{i=1}^{p} \theta_1 = 2\pi\chi(R)$$

where s is the arc length of C_i, and the integral over C_i means the sum of integrals in every regular arc of C_i.

1. 怎麼說它有多彎？　　曹亮吉
2. 高斯曲率到黎曼流形上稱為截曲率。
3. 維數大於 2 的黎曼流形黎曼引入一個抽象且嚴格的方法，就是黎曼曲率張量。

$$R(u,v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$$