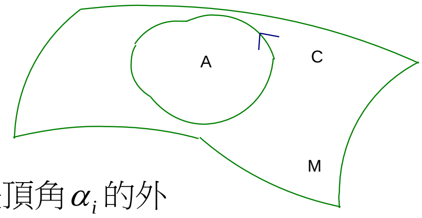
§01 敘述

1. C 是曲面 M 上的 Jordan curve(平滑,封閉,簡單)

   則 $\iint_A KdA + \int_C \kappa_g ds = \int_C d\theta = 2\pi$



2. 若 C 是分段平滑的,A compact,oriented,則

   $\iint_A KdA + \int_C \kappa_g ds + \sum_i (\pi - \alpha_i) = 2\pi\aleph$,其中 $\pi - \alpha_i$ 是頂角 α_i 的外角,$\aleph$ 是 Euler 示性數

§02 預備定理

1. 何謂(1)測地曲率 (2)高斯曲率

   $\frac{dT}{ds} = \kappa_n N + \kappa_g U, U = N \times T$

   則 $\kappa_g = \frac{dT}{ds} \cdot U$ 是為測地曲率

2. Green 定理

   C 是 piecewise smooth Jordan curve

   則 $\iint_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dxdy = \oint_C Pdx + Qdy$

   例 $C: x^2 + \frac{y^1}{4} = 1$,力場 F(x,y)=[3x+y,-x+2y]加諸質點 P,沿 C 逆時針繞一圈所作的功 $W = \oint_C F \cdot dx = \oint Pdx + Qdy = \ldots = -4\pi$

3. 取 lines of curvature 作參數曲線時

   $K = \frac{eg - f^2}{EG - F^2} = \frac{1}{-\sqrt{EG}} \left\{ \frac{\partial}{\partial u} \left(\frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v} \right) \right\}$

4. Liouville's lemma

u-curve,v-curve 在曲面 M 上形成一正交網,$T_P(M)$由 T,U 所張,

假設 T 與 u-curve 夾角為 θ,則 $\begin{cases} T = i_1 \cos\theta + i_2 \sin\theta \\ U = -i_1 \sin\theta + i_2 \cos\theta \end{cases}$, $T \cdot i_1 = \cos\theta$

$ds_1 = ds\cos\theta, ds_2 = ds\sin\theta, ds^2 = ds_1{}^2 + ds_2{}^2$

定義 $\kappa_g = \frac{dT}{ds} \cdot U$

則 $\kappa_g = \kappa_1 \cos\theta + \kappa_2 \sin\theta + \frac{d\theta}{ds}$,其中 $\kappa_1 = (\kappa_g)_{u-curve} = \frac{di_1}{ds_1} \cdot i_2 = -i_1 \cdot \frac{di_2}{ds_1}$

$$\kappa_2 = (\kappa_g)_{v-curve} = \frac{-di_2}{ds_2}\cdot i_1 = i_2 \cdot \frac{di_1}{ds_2}$$

$$\frac{di_1}{ds} = \frac{di_1}{ds_1}\frac{ds_1}{ds} + \frac{di_1}{ds_2}\frac{ds_2}{ds} = \cos\theta\frac{di_1}{ds_1} + \sin\theta\frac{di_1}{ds_2}$$

$$\frac{di_2}{ds} = ... = \cos\theta\frac{di_2}{ds_1} + \sin\theta\frac{di_2}{ds_2}$$

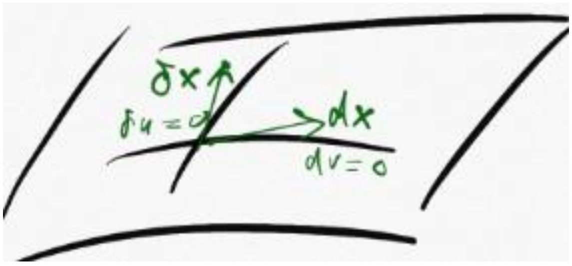
$$U = -i_1\sin\theta + i_2\cos\theta$$

$$\frac{di_1}{ds}\cdot U = \kappa_1\cos^2\theta + \kappa_2\sin\theta\cos\theta, \frac{di_2}{ds}\cdot U = \kappa_1\sin\theta\cos\theta + \kappa_2\sin^2\theta$$

$$\frac{dT}{ds} = \cos\theta\frac{di_1}{ds} + \sin\theta\frac{di_2}{ds} + U\frac{d\theta}{ds}$$

$$\frac{dT}{ds}\cdot U = ... = \kappa_1\cos\theta + \kappa_2\sin\theta + \frac{d\theta}{ds}$$

§03 證明



選一正交坐標系 F=0

$$dA = |dx\times\delta x| = \sqrt{EG-F^2}du\delta v = \sqrt{EG}du\delta v$$

在 u-curve 上,

$$ds_1^{\ 2} = Edu^2 + 2Fdudv + Gdv^2 = Edu^2$$

在 v-curve 上,

$$ds_2^{\ 2} = ... = Gdv^2, ds_2 = \sqrt{G}dv$$

所以 $ds_1 = \sqrt{E}du = ds\cos\theta, ds_2 = \sqrt{G}dv = ds\sin\theta$,由 Liouville's lemma

$$\kappa_g ds = d\theta + \kappa_1\cos\theta ds + \kappa_2\sin\theta ds = d\theta + \kappa_1\sqrt{E}du + \kappa_2\sqrt{G}dv$$

在 u-curve 上, $\frac{dv}{ds} = 0, \frac{du}{ds} = \frac{1}{\sqrt{E}}, \kappa_1 = \Gamma_{11}^2\frac{\sqrt{EG-F^2}}{E\sqrt{E}}, (\Gamma_{11}^2 = \frac{-E_v}{2G}) = -\frac{E_v}{2E\sqrt{G}}$

在 v-curve 上,du=0, $\frac{dv}{ds} = \frac{1}{\sqrt{G}}, \kappa_2 = ... = \frac{G_u}{2G\sqrt{E}}$

$$\int_C \kappa_1\sqrt{E}du + \kappa_2\sqrt{G}dv = \iint_A [\frac{\partial}{\partial u}(\kappa_2\sqrt{G}) - \frac{\partial}{\partial v}(\kappa_1\sqrt{E})]dudv \text{ ……(※)}$$

$$= \iint_A [\frac{\partial}{\partial u}(\frac{G_u}{2\sqrt{EG}}) + \frac{\partial}{\partial v}(\frac{E_v}{2\sqrt{EG}})]dudv$$

$$= \iint_A -K\sqrt{EG}dudv = -\iint_A KdA$$

(F=0 時, $K=-\frac{1}{\sqrt{EG}}\left\{\frac{\partial}{\partial u}(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u})+\frac{\partial}{\partial v}(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v})\right\}$……尚待證明

$$=-\frac{1}{\sqrt{EG}}\left\{\frac{\partial}{\partial u}(\frac{G_u}{2\sqrt{EG}})+\frac{\partial}{\partial v}(\frac{E_v}{2\sqrt{EG}})\right\})$$

所以 當 C 是 M 上一 Jardon Curve

$$\int_C \kappa_g ds=\int_C d\theta-\iint_A KdA$$

$$\iint_A KdA+\int_C \kappa_g ds=\int_C d\theta=2\pi$$

§04 應用

§05 討論

1. 對一個平面上的曲線,K=0,回到切線轉角定理。
2. 半徑=a 的球面,平面 E 到球心的距離=d,和球面截出一圓 C,P 是 C 上一點,求

   $\kappa_g(P)=\ldots\ldots\frac{d}{ar},(r=\sqrt{a^2-d^2})$ 是小圓半徑

$$X(\theta,\varphi)=[a\cos\theta\cos\varphi,a\cos\theta\sin\varphi,a\sin\theta]$$

$$X_\theta=[-a\sin\theta\cos\varphi,-a\sin\theta\sin\varphi,a\cos\theta]$$

$$X_\varphi=[-a\cos\theta\sin\varphi,a\cos\theta\cos\varphi,0]$$

$$E=a^2,F=0,G=a^2\cos^2\theta$$

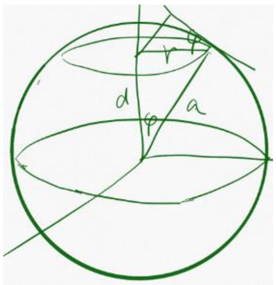
$$dA=\sqrt{EG-F^2}\,d\theta d\varphi=a^2\cos\theta d\theta d\varphi$$

$\iint_A KdA=\int_0^{2\pi}\int_{\theta_0}^{\frac{\pi}{2}}\cos\theta d\theta d\varphi$,其中 $\sin\theta_0=\frac{d}{a}$

$$=2\pi(1-\frac{d}{a})$$

$\int_C \kappa_g ds=2\pi r\kappa_g=2\pi-2\pi(1-\frac{d}{a})=2\pi\frac{d}{a}$,所以 $\kappa_g=\frac{d}{ar}$

another viewpoint



小圓的曲率= $\frac{1}{r}$ 在切平面的投影

$$\kappa_g=\frac{1}{r}\cos\varphi=\frac{1}{r}\times\frac{d}{a}=\frac{d}{ar}$$