§ 1 曲面 M:X=X[u,v]，以下建立么速測地坐標系，

$X_u = \dfrac{\partial X}{\partial u}, X_v = \dfrac{\partial X}{\partial v}$ ，使得 E=1，F=0， $ds^2 = du^2 + Gdv^2$

取 $e_1 = X_u, e_2 = X_v / g, e_3 = \dfrac{X_u \times X_v}{g} = N$ ， $g^2 = G = X_v \cdot X_v$

$(|e_2| = \left|\dfrac{X_v \cdot X_v}{g^2}\right| = \dfrac{G}{g^2} = 1)$

$$K = \kappa_1\kappa_2 = \frac{N_u \times N_v}{X_u \times X_v} = \frac{\det(b_{ij})}{\det(g_{ij})} = \frac{eg - f^2}{EG - F^2}$$

$$N_u = <N_u, e_1> e_1 + <N_u, e_2> e_2$$

$$N_v = <N_v, e_1> e_1 + <N_v, e_2> e_2$$

$$N_u \times N_v = \{<N_u, e_1><N_v, e_2> - <N_u, e_2><N_v, e_1>\} e_1 \times e_2$$

$$= \frac{1}{g^2}\{<N_u, X_u><N_v, X_v> - <N_u, X_v><N_v, X_u>\} X_u \times X_v$$

$$= \frac{1}{g^2}\{<N, X_{uu}><N, X_{vv}> - <N, X_{uv}>^2\} X_u \times X_v$$

（由 $N \cdot X_u = 0$ ，得 $N_u \cdot X_u = -N \cdot X_{uu}$ 。由 $N \cdot X_v = 0$ ，得 $N_v \cdot X_v = -N \cdot X_{vv}$

同理 $N_u \cdot X_v = N_v \cdot X_u = -N \cdot X_{uv}$ ）

由 $\dfrac{\partial}{\partial u} X_{uv} - \dfrac{\partial}{\partial v} X_{uu} = 0$ ，兩邊對 X_v 作內積

$$<\frac{\partial}{\partial u} X_{uv}, X_v> - <\frac{\partial}{\partial v} X_{uu}, X_v> = 0$$

（ $\dfrac{\partial}{\partial u} <X_{uv}, X_v> = <\dfrac{\partial}{\partial u} X_{uv}, X_v> + <X_{uv}, X_{uv}>$ ）

$$\frac{\partial}{\partial u} <X_{uv}, X_v> - <X_{uv}, X_{uv}> - \frac{\partial}{\partial v}(X_{uu}, X_v) + <X_{uu}, X_{vv}> = 0 \text{ ......（※）}$$

$$g^2 = X_v \cdot X_v$$

$\dfrac{\partial}{\partial u} g^2 = X_{uv} \cdot X_v + X_v \cdot X_{uv}$ ，所以 $X_{uv} \cdot X_v = \dfrac{1}{2}\dfrac{\partial}{\partial u} g^2 = gg_u$

$$\frac{\partial}{\partial u}<X_{uv},X_v>=\frac{\partial}{\partial u}gg_u=g_u{}^2+gg_{uu}\ldots(1)$$

$$X_{uv}=<X_{uv},X_u>X_u+<X_{uv},X_v>\frac{X_v}{g^2}+<X_{uv},N>N$$

$$-<X_{uv},X_{uv}>=-\frac{1}{g^2}<X_{uv},X_v>^2-<X_{uv},N>^2\ldots(2)$$

(因為 $g^2=X_v\cdot X_v,\frac{\partial}{\partial u}g^2=X_{uv}\cdot X_v+X_v\cdot X_{uv}=2X_{uv}\cdot X_v$)

$$=-\frac{1}{g^2}(\frac{\partial}{\partial u}(\frac{1}{2}g^2))^2-<X_{uv},N>^2=-g_u{}^2-<X_{uv},N>^2$$

$$\frac{\partial}{\partial v}<X_{uu},X_v>=0\ldots(3)$$

(因為 $X_{uu}=\_\_X_u+\_\_X_v+\_\_N=\_\_N$ ，$<X_v,N>=0$)

X_{uu} 只有法部，所以 $X_{uu}=<X_{uu},N>N$

$$<X_{uu},X_{vv}>=<<X_{uu},N>N,X_{vv}>=<X_{uu},N><X_{vv},N>\ldots(4)$$

(※)變成

$$gg_{uu}-<X_{uv},N>+<X_{uu},N><X_{vv},N>=0$$

其中 $-<X_{uv},N>^2+<X_{uu},N><X_{vv},N>=g^2K$

所以 $K=\frac{-g_{uu}}{g}$

[習作]

若只有條件 F=0，則 $K=-\frac{1}{2g}\{(\frac{E_v}{g})_v+(\frac{G_u}{g})_u\}$ ，$g=\sqrt{EG}$

§2 共變微分(covariant derivative)

向量場 V 沿 Y 的共變微分 $\nabla_Y V=(dV(Y))^T$

$\alpha=\alpha(t),\alpha(0)=Y$ ，$X=(u'(0),v'(0))$

$$\frac{d\alpha}{dt}(0)=X \text{ ，} dV(X)=\frac{d(V\circ\alpha(t))}{dt}(0)=\frac{dV}{du}\frac{du}{dt}+\frac{dV}{dv}\frac{dv}{dt}=<(\frac{\partial V}{\partial u},\frac{\partial V}{\partial v}>,X>$$

［習作］

赤道 $\gamma(t)=(\cos t,\sin t,0)$

向量場 $V=\frac{1}{5}(-\sin t,\cos t,0)+\frac{1}{5}(0,0,1)=\frac{1}{5}(-\sin t,\cos t,1)$

證明 $\nabla_{\frac{d\gamma}{dt}}V=(\frac{dV}{dt})^T=0$（即 V 是球面上沿 γ 的"平行"向量場）

取 一平行向量場 $Z=\cos\alpha e_1+\sin\alpha e_2$ ，$(\frac{dZ}{dt})^T=0$

證明 $\frac{d\alpha}{dt}=-g_u\frac{dv}{dt}$

$$\frac{dZ}{dt}=(-\sin\alpha e_1+\cos\alpha e_2)\frac{d\alpha}{dt}+(\cos\alpha\frac{de_1}{dt}+\sin\alpha\frac{de_2}{dt})$$

$0=\frac{dZ}{dt}\cdot e_2=\cos\alpha\frac{d\alpha}{dt}+\cos\alpha(\frac{de_1}{dt}\cdot e_2)$ ，所以

$$\frac{d\alpha}{dt}=-\frac{de_1}{dt}\cdot e_2=-(\frac{\partial X_u}{\partial u}\frac{du}{dt}+\frac{\partial X_u}{\partial v}\frac{dv}{dt})\cdot\frac{X_v}{g}$$

$$=-<X_{uu},\frac{X_v}{g}>\frac{du}{dt}-<X_{uv},\frac{X_v}{g}>\frac{dv}{dt}$$

$$=0-X_{uv}\cdot X_v\frac{1}{g}\frac{dv}{dt}\quad (X_{uv}\cdot X_v=\frac{\partial}{\partial u}(\frac{1}{2}g^2))$$

$$=-g_u\frac{dv}{dt}$$

§ 3

一 simple,封閉曲線 $\gamma(t),a\le t\le b,\gamma(a)=\gamma(b)$

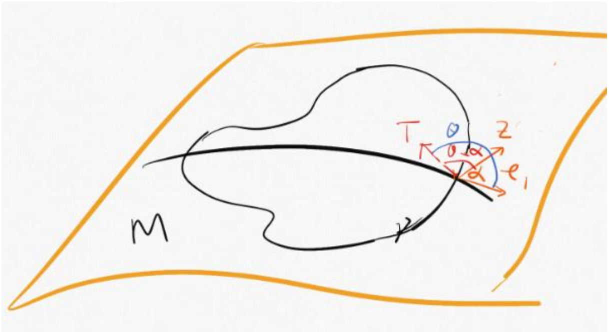
holonomy 角 $\delta_\gamma\alpha=\int_a^b(\frac{d\alpha}{dt})dt=\alpha(b)-\alpha(a)$ 是角總轉量

Green 定理 $\int_\gamma Pdu+Qdv=\iint_A(\frac{\partial Q}{\partial u}-\frac{\partial P}{\partial v})dudv$

$$\delta_\gamma\alpha=\int_a^b(\frac{d\alpha}{dt})dt=\int_a^b(-g_u\frac{dv}{dt})dt=\int_\gamma -g_u dv=\iint_A -g_{uu}dudv$$

$$=\iint_A\frac{-g_{uu}}{g}d\sigma=\iint_A Kd\sigma\text{（因為 }d\sigma=gdudv\text{）}$$

§ 4 $T=\dfrac{d\lambda}{ds}=\cos\theta e_1+\sin\theta e_2$



由切線轉角定理$\delta_\gamma\theta=2\pi$

沿γ的平行向量場

$Z=\cos\alpha e_1+\sin\alpha e_2$

$\theta=\alpha+(\theta-\alpha)$

$$2\pi=\delta_\gamma\theta=\delta_\gamma\alpha+\delta_\gamma(\theta-\alpha)=\iint_A Kd\sigma+\int_\gamma\frac{d}{ds}(\theta-\alpha)ds$$

以下證明$\dfrac{d}{ds}(\theta-\alpha)=\kappa_g$

$T=\cos\theta e_1+\sin\theta e_2, N=-\sin\theta e_1+\cos\theta e_2$

$$\frac{dT}{ds}=(-\sin\theta e_1+\cos\theta e_2)\frac{d\theta}{ds}+\cos\theta\frac{de_1}{ds}+\sin\theta\frac{de_2}{ds}$$

$$\kappa_g=(\frac{dT}{ds})^T=\frac{dT}{ds}\cdot N$$

$$=(\sin^2\theta+\cos^2\theta)\frac{d\theta}{ds}+(-\sin^2\theta\frac{de_2}{ds}\cdot e_1+\cos^2\theta\frac{de_1}{ds}\cdot e_2)$$

$$=\frac{d\theta}{ds}+\frac{de_1}{ds}\cdot e_2=\frac{d\theta}{ds}-\frac{d\alpha}{ds}=\frac{d}{ds}(\theta-\alpha)$$

得$\displaystyle\iint_A Kd\sigma+\int_\gamma\kappa_g ds=2\pi$