Do Carmo 習作 4-3 (1)
X 是正交參數系，即 F=0

試證 $K=-\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v+\left(\frac{G_u}{\sqrt{EG}}\right)_u\right\}$

(即 $eg-f^2=-\frac{\sqrt{EG}}{2}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v+\left(\frac{G_u}{\sqrt{EG}}\right)_u\right\}$ ......(A)

(一) $[ij,k]=X_{ij}\cdot X_k=\frac{1}{2}\{\frac{\partial g_{jk}}{\partial u^i}+\frac{\partial g_{ik}}{\partial u^j}-\frac{\partial g_{ij}}{\partial u^k}\}$

(二) $\Gamma_{ij}^k=[ij,l]g^{lk}$，其中 $g_{ij}g^{jk}=\delta_i^k$

(三) $X_{ij}=\Gamma_{ij}^k X_k+b_{ij}N$

$[11,1]=\frac{1}{2}E_u,[11,2]=\frac{1}{2}(2F_u-E_v),[12,1]=\frac{1}{2}E_v$

$[12,2]=\frac{1}{2}G_u,[22,1]=\frac{1}{2}\{2F_v-G_u\},[22,2]=\frac{1}{2}G_v$

$\Gamma_{11}^1=(GE_u-2FF_u+FE_v)/2\Delta=\frac{E_u}{2E}$ ，$\Delta=EG-F^2=EG$

$\Gamma_{11}^2=(2EF_u-EE_v-FE_u)/2\Delta=\frac{-E_v}{2G}$

$\Gamma_{12}^1=(GE_v-FG_u)/2\Delta=\frac{E_v}{2E}$

$\Gamma_{12}^2=(EG_u-FE_v)/2\Delta=\frac{G_u}{2G}$

$\Gamma_{22}^1=(2GF_v-GG_u-FG_v)/2\Delta=\frac{-G_u}{2E}$

$\Gamma_{22}^2=(EG_v-2FF_v+FG_u)/2\Delta=\frac{G_v}{2G}$

$X_{uu}\cdot X_v=\Gamma_{11}^2G=\frac{-E_v}{2},X_{uv}\cdot X_v=\Gamma_{12}^2G=\frac{G_u}{2},X_{vv}\cdot X_v=\Gamma_{22}^2G=\frac{G_v}{2}$

$N_u\cdot X_v=-f,N_v\cdot X_v=-g$

$(X_{uv})_u=(X_{uu})_v$

$X_{uv}=\Gamma_{12}^1X_u+\Gamma_{12}^2X_v+fN$ ，$X_{uu}=\Gamma_{11}^1X_u+\Gamma_{11}^2X_v+eN$

$(X_{uv})_u=(\frac{E_v}{2E})_uX_u+(\frac{E_v}{2E})X_{uu}+(\frac{G_u}{2G})_uX_v+(\frac{G_u}{2G})X_{uv}+f_uN+fN_u$

$$(X_{uu})_v = (\frac{E_u}{2E})_v X_u + (\frac{E_u}{2E}) X_{uv} + (\frac{-E_v}{2G})_v X_v + (\frac{-E_v}{2G}) X_{vv} + e_v N + eN_v$$

兩邊對 X_v 作內積

$$(\frac{E_v}{2E})(-\frac{E_v}{2}) + (\frac{G_u}{2G})_u G + (\frac{G_u}{2G})\frac{G_u}{2} - f^2$$

$$= (\frac{E_u}{2E})\frac{G_u}{2} + (\frac{-E_v}{2G})_v G + (\frac{-E_v}{2G})\frac{G_v}{2} - eg$$

$$eg - f^2 = (\frac{E_u}{2E})\frac{G_u}{2} + (\frac{-E_v}{2G})_v G + (\frac{-E_v}{2G})\frac{G_v}{2} - (\frac{E_v}{2E})(\frac{-E_v}{2}) - (\frac{G_u}{2G})_u G - (\frac{G_u}{2G})\frac{G_u}{2}$$

$$= \frac{E_u G_u}{4E} - \frac{1}{2}(\frac{E_v}{G})_v G - \frac{E_v G_v}{4G} + \frac{(E_v)^2}{4E} - \frac{1}{2}(\frac{G_u}{G})_u G - \frac{(G_u)^2}{4G}$$

$$= \{\frac{E_u G_u}{4E} + \frac{(G_u)^2}{4G} - \frac{G_{uu}}{2}\} + \{\frac{E_v G_v}{4G} + \frac{(E_v)^2}{4E} - \frac{E_{vv}}{2}\} \text{......(B)}$$

現在暫時只能這樣了，把(A)展開 證明它等於(B)

$$-\frac{\sqrt{EG}}{2}(\frac{G_u}{\sqrt{EG}})_u = -\frac{\sqrt{EG}}{2}\{\frac{\sqrt{EG}G_{uu} - (\sqrt{EG})_u G_u}{EG}\} = \ldots = \frac{E_u G_u}{4E} + \frac{(G_u)^2}{4G} - \frac{G_{uu}}{2}$$

$$-\frac{\sqrt{EG}}{2}(\frac{E_v}{\sqrt{EG}})_v = -\frac{\sqrt{EG}}{2}\{\frac{\sqrt{EG}E_{vv} - (\sqrt{EG})_v E_v}{EG} = \ldots\ldots = \frac{E_v G_v}{4G} + \frac{(E_v)^2}{4E} - \frac{E_{vv}}{2}$$

註

1.若取么速 lines of curvature 作參數曲線，則 E=1，F=f=0，則 K=-g_{uu}/g，其中 G=g^2

$$K = -\frac{1}{\sqrt{EG}}\left\{\frac{\partial}{\partial u}(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}) + \frac{\partial}{\partial v}(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v})\right\}$$

2.

3.黎曼曲率張量 $R_{ijkl} = R^m_{ijk} g_{ml}$ ，則 $K = \frac{R_{1212}}{EG - F^2}$

$$R_{1212} = R^n_{121} g_{2n} = g^{n2} g_{2n}(eg - f^2) = eg - f^2$$