Beltrami-Enneper 定理

Theorem 4. *The absolute value of the torsion τ at a point of an asymptotic curve whose curvature is nowhere zero is given by*

$$|\tau| = \sqrt{-K}$$

where K is the Gaussian curvature of the surface at a given point.

假設通過 P 點的 asymptotic curve 為$\alpha(t)$，且$\alpha(0)=P$

在 P 點的 Frenet 標架為$\{e_1,e_2,e_3\}$，其中$e_1=t, e_2=n, e_3=b$

此時 osculating plane 與切平面是重合，所以$\{t,n\}$所張的平面即切平面，因此$b=t\times n$為曲面的法向量 N(surface normal)

(因為$\dfrac{dt}{ds}=\kappa_n N+\kappa_g Y=\kappa n$，在 asymptotic direction $\kappa_n=0$，所以 n 落在切平面上。)

$$N' = \frac{db}{ds} = -\tau n$$

$$N'(0) = dN_p(\alpha'(0)) = dN_p(t) = dN_p(e_1\cos\theta + e_2\sin\theta)$$

$$= \cos\theta\, dN_p(e_1) + \sin\theta\, dN_p(e_2) = -\kappa_1\cos\theta\, e_1 - \kappa_2\sin\theta\, e_2$$

$$\tau^2 = |N'|^2 = \kappa_1^2\cos^2\theta + \kappa_2^2\sin^2\theta$$

由 Euler 公式與 asymptotic curve 的定義

$$\kappa_1\cos^2\theta + \kappa_2\sin^2\theta = 0$$

解出$\cos^2\theta = \dfrac{-\kappa_2}{\kappa_1-\kappa_2}, \sin^2\theta = \dfrac{\kappa_1}{\kappa_1-\kappa_2}$

$$\tau^2 = -\kappa_1\kappa_2 = -K$$