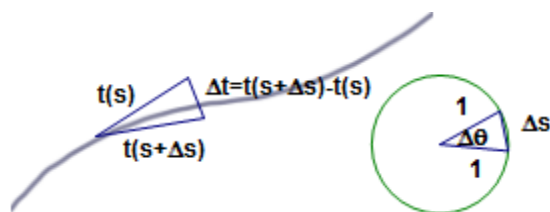


§ 古典微分幾何中的曲率

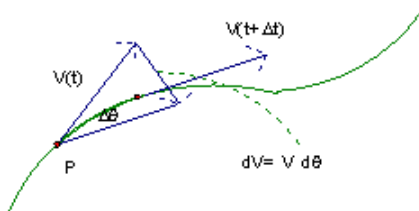


$$\text{曲率} = \kappa = |\kappa| = \frac{d\theta}{ds}$$

$$|t(s + \Delta s) - t(s)| \approx \Delta s = \Delta \theta$$

§ 01 曲率向量

$$\kappa = \lim_{\Delta s \rightarrow 0} \frac{t(s + \Delta s) - t(s)}{\Delta s} = \frac{dt}{ds} \quad (\text{單位切向量對弧長的變化率})$$

§02 一質點沿一曲線 $\alpha(t)$ 作等速率運動, 則加速度

$$|\vec{a}| = \lim_{\Delta t \rightarrow 0} \frac{V(t + \Delta t) - V(t)}{\Delta t} = \frac{dV}{dt} = \frac{DV}{ds} \cdot \frac{ds}{dt} = |V| \frac{d\theta}{ds} \cdot \frac{ds}{dt} = \kappa V^2$$

說明

$$(1) s(t) = \int_{t_0}^t \left| \frac{d\alpha}{dt} \right| dt, \text{ 則 } \frac{ds}{dt} = \left| \frac{d\alpha}{dt} \right| = V(t)$$

$$(2) \text{ 對等速圓周運動, 因為 } \kappa = \frac{1}{R}, \text{ 所以 } |\vec{a}| = \frac{V^2}{R}$$

$$X' = \frac{dX}{ds}, \dot{X} = \frac{dX}{dt}, \text{ 則 } \kappa^2 = X'' \cdot X''$$

$$\text{因為 } X' \perp X'' \text{ 且 } |X'| = 1, \text{ 所以 } \kappa = |X' \wedge X''| = \frac{|\dot{X} \wedge \ddot{X}|}{|\dot{X}|^3}$$

(證明在最後面)

◇習作◇

1. 求 $y=f(x)$ 的曲率函數
2. 求螺線 (helix) $X=(a\cos t, a\sin t, bt)$ 的曲率
3. 橢圓 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ 的曲率函數
4. $\alpha(t) = (r\cos t, r\sin t, 0)$, 求 $\int_{\alpha} \kappa ds =$

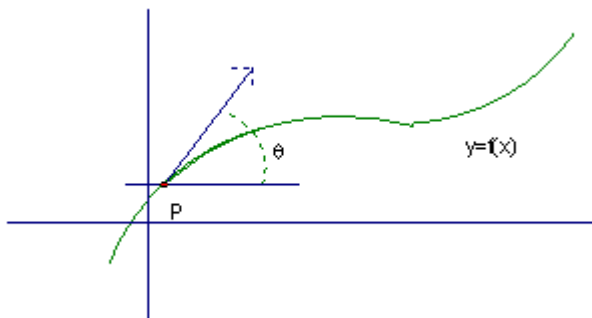
◇解答◇

1. 求 $y=f(x)$ 的曲率函數

弧長 $ds^2 = dx^2 + dy^2$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + (f'(x))^2}$$

$$\text{所以 } \frac{dx}{ds} = \frac{1}{\sqrt{1 + (f'(x))^2}} \dots\dots (1)$$



$$f'(x) = \tan \theta, \text{ 所以 } f''(x) = \frac{d \tan \theta}{d\theta} \cdot \frac{d\theta}{dx} = \sec^2 \theta \cdot \frac{d\theta}{dx} = (1 + (f'(x))^2) \frac{d\theta}{dx}$$

$$\frac{d\theta}{dx} = \frac{f''(x)}{1 + (f'(x))^2} \dots\dots (2), \text{ 由 (1) (2)}$$

$$\kappa = \frac{d\theta}{ds} = \frac{d\theta}{dx} \cdot \frac{dx}{ds} = \frac{f''(x)}{[1 + (f'(x))^2]^{\frac{3}{2}}}$$

如果用弧長做參數，在反曲點 $f'(s) = 0, f''(s) = 0$ ，則 $\kappa = |f''(s)| = 0$

$$2. \kappa = \frac{a}{a^2 + b^2}$$

$$3. X = (a \cos \theta, b \sin \theta), \dot{X} = (-a \sin \theta, b \cos \theta), \ddot{X} = (-a \cos \theta, -b \sin \theta)$$

$$|\dot{X} \wedge \ddot{X}| = ab, \text{ 所以 } \kappa^2 = \frac{|\dot{X} \times \ddot{X}|^2}{(\dot{X} \cdot \dot{X})^3}, \kappa = \frac{ab}{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{\frac{3}{2}}}$$

$$\text{在短軸端點 } \theta = \frac{\pi}{2}, \kappa = \frac{b}{a^2}$$

$$f(\theta) = a^2 \sin^2 \theta + b^2 \cos^2 \theta = b^2 + (a^2 - b^2) \sin^2 \theta \text{ 在 } \theta = 0, \frac{\pi}{2} \text{ 產生極值}$$

所以橢圓的“頂點” ($\frac{d\kappa}{ds} = 0$ 的地方) 在四個端點產生。

這是四頂點定理的一個例子。

$$4. \alpha(t) = (r \cos t, r \sin t, 0), \text{ 求 } \int_{\alpha} \kappa ds =$$

$$s = s(t) = \int_0^t |\dot{\alpha}(t)| dt = rt, \text{ 所以 } \alpha(s) = (r \cos \frac{s}{r}, r \sin \frac{s}{r}, 0)$$

$$t(s) = \frac{d\alpha}{ds} = (-\sin \frac{s}{r}, \cos \frac{s}{r}, 0)$$

$$t'(s) = (-\frac{1}{r} \cos \frac{s}{r}, -\frac{1}{r} \sin \frac{s}{r}, 0) = \frac{1}{r} n, \text{ 其中 } n = (-\cos \frac{s}{r}, -\sin \frac{s}{r}, 0), \text{ 所以 } \kappa = \frac{1}{r}$$

$$\int_{\alpha} \kappa ds = \int_0^{2\pi} \frac{1}{r} ds = 2\pi, \text{ 這是切線轉角定理。}$$

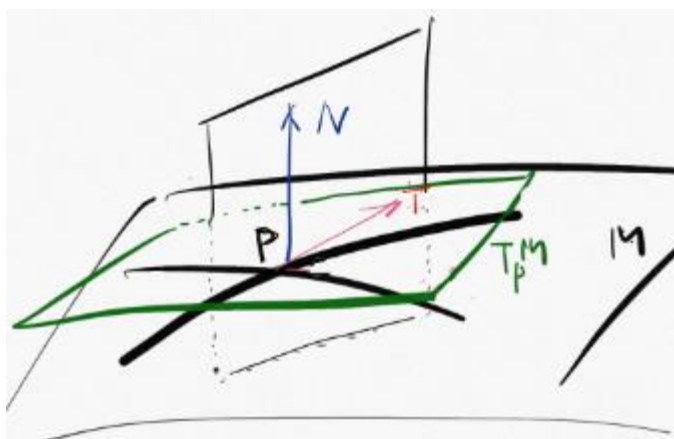
$$\frac{ds}{dt} = (\dot{X} \cdot \dot{X})^{\frac{1}{2}}, \frac{dt}{ds} = (\dot{X} \cdot \dot{X})^{-\frac{1}{2}}, \frac{d^2 t}{ds^2} = \frac{d}{dt} (\dot{X} \cdot \dot{X})^{-\frac{1}{2}} \frac{dt}{ds} = \dots = -(\dot{X} \cdot \dot{X})^{-2} (\dot{X} \cdot \ddot{X})$$

$$X' = \frac{dX}{ds} = \dot{X} \frac{dt}{ds} = \dot{X} (\dot{X} \cdot \dot{X})^{-\frac{1}{2}}$$

$$X'' = \ddot{X} \left(\frac{dt}{ds}\right)^2 + \dot{X} \frac{d^2 t}{ds^2} = \dots = \frac{(\dot{X} \cdot \dot{X}) \ddot{X} - (\dot{X} \cdot \ddot{X}) \dot{X}}{(\dot{X} \cdot \dot{X})^2}$$

$$\text{(注意到 } |\dot{X} \wedge \dot{X}| = 0, (\dot{X} \cdot \dot{X})^{\frac{1}{2}} = |\dot{X}|, \text{ Then } \kappa = |X' \wedge X''| = \frac{|\dot{X} \wedge \ddot{X}|}{|\dot{X}|^3}$$

§ 03 曲面上的曲線



$$\kappa = \frac{dT}{ds} = \kappa_n N + \kappa_g U$$

U 落在切平面 $T_p M$ 上, spanned by X_u, X_v

$$\kappa_n = \frac{dT}{ds} \cdot N = -\frac{dN}{ds} \cdot T = \frac{II}{I}$$

$$\rightarrow \begin{cases} H = \frac{1}{2}(\kappa_1 + \kappa_2) \\ K = \kappa_1 \kappa_2 = \frac{eg - f^2}{EG - F^2} \end{cases}$$

取 lines of curvature 作參數座標時, $F=f=0$

$$\kappa_1 = \frac{e}{E} \dots (dv=0), \kappa_2 = \frac{g}{G} \dots (du=0)$$

$$\kappa_g = \frac{dT}{ds} \cdot U, U = N \times T \rightarrow \text{測地線} \langle \frac{dT}{ds}, X_k \rangle = 0, \text{得 } u^{k''} + \Gamma_{ij}^k u^i u^j = 0$$

κ_n 的最大 最小值稱為主法曲率(principal normal curvature)

N 是曲面 M 上過 P 點的單位法向量

過 P 的垂直斷面所截的曲線 X

在 P 點的單位切向量為 T

$$T = \frac{dX}{ds}$$

κ_n 是定義在 compact set 上的連續函數, 所以其極值必存在

$$\kappa_n = \frac{II}{I} = \frac{edu^2 + 2fdudv + gdv^2}{Edu^2 + 2Fdudv + Gdv^2} = \frac{e + 2f\lambda + g\lambda^2}{E + 2F\lambda + G\lambda^2}, \text{其中 } \lambda = \frac{dv}{du}$$

$$\frac{d\kappa_n}{d\lambda} = 0 \text{ 時}$$

$$(E + 2F\lambda + G\lambda^2)(2f + 2g\lambda) - (e + 2f\lambda + g\lambda^2)(2F + 2G\lambda) = 0$$

$$\frac{f + g\lambda}{F + G\lambda} = \frac{e + 2f\lambda + g\lambda^2}{E + 2F\lambda + G\lambda^2} = \frac{(e + f\lambda) + \lambda(f + g\lambda)}{(E + F\lambda) + \lambda(F + G\lambda)} = \frac{e + f\lambda}{E + F\lambda} \text{ (合分比)}$$

$$(fG - gF)\lambda^2 + (eG - gE)\lambda + (eF - fE) = 0$$

$$(fG - gF)dv^2 + (eG - gE)dudv + (eF - fE)du^2 = 0 \dots \#$$

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0 \dots\dots \text{此兩方向 } \frac{dv}{du}, \kappa_n \text{ 有極值}$$

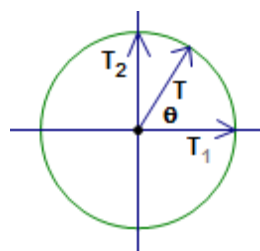
由#可證 principal direction 互相垂直,其積分曲線稱為 lines of curvature
pf

$$dX \perp \delta X \Leftrightarrow Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v = 0$$

$$\text{假設 } \lambda_1 = \frac{dv}{du}, \lambda_2 = \frac{\delta v}{\delta u}, \text{則 } E + F(\lambda_1 + \lambda_2) + G\lambda_1\lambda_2 = 0$$

$$\begin{vmatrix} \lambda^2 & -\lambda & 1 \\ E & F & G \\ e & f & g \end{vmatrix} = 0 \text{ 的兩根為 } \lambda_1, \lambda_2, \lambda_1 + \lambda_2 = \frac{eG - Eg}{Fg - fG}, \lambda_1\lambda_2 = \frac{Ef - eF}{Fg - fG}$$

則 $E + F(\lambda_1 + \lambda_2) + G\lambda_1\lambda_2 = 0$, 得證



$$T = \cos \theta T_1 + \sin \theta T_2$$

$$\text{則 } \kappa_n(\theta) = \cos^2 \theta \cdot \kappa_1 + \sin^2 \theta \cdot \kappa_2 \dots\dots \text{Euler 定理 1760}$$

證 1

取 lines of curvature 作參數曲線 則 $F=0$

在 u-curve 上, $du=0$, 在 v-curve 上, $dv=0$

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0, \text{所以 } \begin{cases} Ef - eF = 0 \\ gF - fG = 0 \end{cases}, \text{但 } EG - F^2 > 0, EG > 0, \text{所以 } f=0$$

$$\kappa_n = \frac{edu^2 + gdv^2}{Edu^2 + Gdv^2} = e\left(\frac{du}{ds}\right)^2 + g\left(\frac{dv}{ds}\right)^2$$

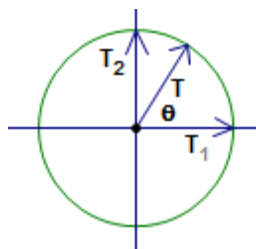
$$\kappa_1 = \frac{e}{E} (\delta v = 0), \kappa_2 = \frac{g}{G} (du = 0)$$

$$\begin{aligned} \cos \theta &= \frac{dx \cdot \delta x}{|dx| |\delta x|} = \frac{Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v}{ds \sqrt{E\delta u^2 + 2F\delta u\delta v + G\delta v^2}} \\ &= \frac{Edu\delta u}{ds \sqrt{E\delta u^2}} = \sqrt{E} \frac{du}{ds} \end{aligned}$$

$$\sin \theta = \sqrt{1 - \cos^2 \theta} = \dots = \sqrt{G} \frac{dv}{ds}$$

$$\kappa_n = e \left(\frac{du}{ds} \right)^2 + g \left(\frac{dv}{ds} \right)^2 = \frac{e}{E} \cdot E \left(\frac{du}{ds} \right)^2 + \frac{g}{G} \cdot G \left(\frac{dv}{ds} \right)^2 = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta$$

$$\frac{1}{2\pi} \int_0^{2\pi} \kappa_n(T) d\theta = \frac{1}{2\pi} \int_0^{2\pi} (\kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta) d\theta = \frac{\kappa_1 + \kappa_2}{2} \text{ 這是均曲率的意義。}$$



Euler 公式(證 2)

$A = -de_3$ (即 do Carmo 的 dN_p) 是 self-adjoint

即 $\langle Av, w \rangle = \langle v, Aw \rangle$

當 $|T|=1$ 時, $\kappa_n(T) = II(T, T) = \langle AT, T \rangle$

T_1, T_2 是兩主法向量, 則 $\langle AT_1, T_1 \rangle = \kappa_1, \langle AT_2, T_2 \rangle = \kappa_2$

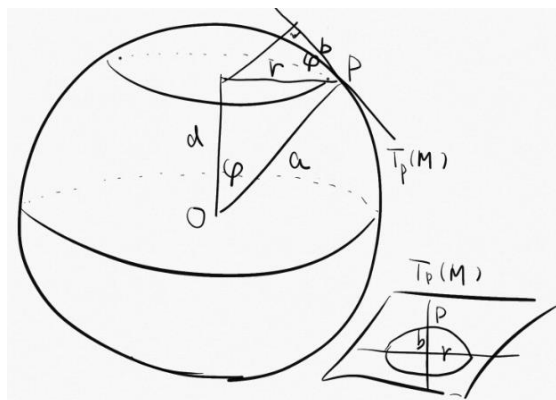
$$T = \cos \theta T_1 + \sin \theta T_2$$

$$\kappa_n = \langle AT, T \rangle = \langle A(\cos \theta T_1 + \sin \theta T_2), \cos \theta T_1 + \sin \theta T_2 \rangle$$

$$= \langle \cos \theta (AT_1) + \sin \theta (AT_2), \cos \theta T_1 + \sin \theta T_2 \rangle$$

$$= \cos^2 \theta \langle AT_1, T_1 \rangle + \sin^2 \theta \langle AT_2, T_2 \rangle$$

$$= \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta$$



半徑=a 的球面上 小圓半徑 C=r

求在 P 點的曲率

小圓 C 在切平面的投影為一橢圓

$$\kappa_g(P) = \frac{b}{r^2} = \frac{r \cos \varphi}{r^2} = \frac{d}{ar}$$

[看 curvature01][Gauss-Bonnet]

$$\kappa = \frac{1}{r} \cos \varphi = \frac{d}{ar} \text{ 這是 Meusnier 定理}$$

高斯曲率

$$X \text{ 是正交參數系，即 } F=0, \text{ 則 } K = -\frac{1}{2\sqrt{EG}} \left\{ \left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right\}$$

Liouville

$$\left[\frac{Dw}{dt} \right] = \frac{1}{2\sqrt{EG}} \left\{ G_u \frac{dv}{dt} - E_v \frac{du}{dt} \right\} + \frac{d\varphi}{dt}$$

$$\text{所以 } \kappa_g = \frac{1}{2\sqrt{EG}} \left\{ G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right\} + \frac{d\varphi}{ds}$$

$(\kappa_g)_1$ 是 u-curve 的 geodesic, $v=\text{const}$ 則

$$(1) \frac{dv}{ds} = 0 \quad (2) ds^2 = Edu^2, \quad \therefore \frac{du}{ds} = \frac{1}{\sqrt{E}}$$

$$(\kappa_g)_1 = \frac{1}{2\sqrt{EG}} \times E_v \times \frac{1}{\sqrt{E}} = -\frac{E_v}{2E\sqrt{G}}$$

$$\text{同理 } (\kappa_g)_2 = \frac{G_u}{2G\sqrt{E}}$$

$$\kappa_g = \frac{1}{2\sqrt{EG}} \left\{ -E_v \frac{du}{ds} + G_u \frac{dv}{ds} \right\} + \frac{d\varphi}{ds} \dots (*)$$

$$(e_1 = \frac{X_u}{\sqrt{E}}, e_2 = \frac{X_v}{\sqrt{G}}, \quad \alpha' = X_u \frac{du}{ds} + X_v \frac{dv}{ds} = e_1 \cos \varphi + e_2 \sin \varphi)$$

$$\sqrt{E} \frac{du}{ds} = \cos \varphi, \quad \sqrt{G} \frac{dv}{ds} = \sin \varphi \quad \text{所以 } \kappa_g = (\kappa_g)_1 \cos \varphi + (\kappa_g)_2 \sin \varphi + \frac{d\varphi}{ds}$$

$$\text{其中 } (\kappa_g)_1 \cos \varphi = \frac{-E_v}{2E\sqrt{G}} \cdot \sqrt{E} \frac{du}{ds} = \frac{-E_v}{2\sqrt{EG}} \frac{du}{ds}$$

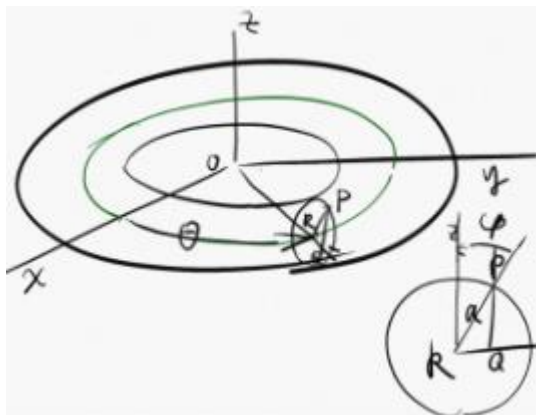
$$(\kappa_g)_2 \sin \varphi = \dots = \frac{G_u}{2\sqrt{EG}} \frac{dv}{ds}$$

$$\int_C (\kappa_g)_1 \cos \varphi ds = \int_C \left(\frac{-E_v}{2\sqrt{EG}} \right) du = \iint_A \frac{\partial}{\partial v} \left(\frac{E_v}{2\sqrt{EG}} \right) dudv$$

$$\text{同理 } \int_C (\kappa_g)_2 \sin \varphi ds = \int_C \frac{G_u}{2\sqrt{EG}} dv = -\iint_A \frac{\partial}{\partial u} \left(\frac{G_u}{2\sqrt{EG}} \right) dudv$$

所以 最後還是這個問題

$$\frac{\partial}{\partial v} \left(\frac{E_v}{\sqrt{EG}} \right) = \frac{1}{\sqrt{EG}} \left(\frac{E_v}{\sqrt{EG}} \right)$$



輪胎面(torus)的高斯曲率

管心曲線為 $y(\theta) = [b \cos \theta, b \sin \theta, 0]$

$$\overline{OQ} = \overline{OR} + \overline{RQ} = b + a \sin \varphi$$

$$X(\theta, \varphi) = [(b + a \sin \varphi) \cos \theta, (b + a \sin \varphi) \sin \theta, a \cos \varphi]$$

$$X_\theta = [-(b + a \sin \varphi) \sin \theta, (b + a \sin \varphi) \cos \theta, 0]$$

$$X_\varphi = [a \cos \varphi \cos \theta, a \cos \varphi \sin \theta, -a \sin \varphi]$$

$$E = (b + a \sin \varphi)^2, F = 0, G = a^2$$

$$X_\theta \times X_\varphi = a(b + a \sin \varphi)[- \cos \theta \sin \varphi, - \sin \theta \sin \varphi, - \cos \varphi]$$

所以 $N = [- \cos \theta \sin \varphi, - \sin \theta \sin \varphi, - \cos \varphi]$

$$X_{\theta\theta} = \dots, X_{\theta\varphi} = \dots, X_{\varphi\varphi} = \dots$$

$$e = X_{\theta\theta} \cdot N = (b + a \sin \varphi) \sin \varphi$$

$$f = X_{\theta\varphi} \cdot N = 0$$

$$g = X_{\varphi\varphi} \cdot N = a^2$$

$$K = \frac{eg - f^2}{EG - F^2} = \frac{a(b + a \sin \varphi) \sin \varphi}{a^2 (b + a \sin \varphi)^2} = \frac{\sin \varphi}{a(b + a \sin \varphi)}$$