



The catenoid is the only minimal surface of revolution .

懸鍊線  $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$  繞  $x$  軸一圈，得懸鍊面 (catenoid) 是一個極小曲面。

預備知識

$$1. \sinh x = \frac{1}{2}(e^x - e^{-x}), \cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\text{若 } y = \cosh^{-1} x, \text{ 則 } \frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}}$$

$$2. \text{ 旋轉面 } z = f(x) \text{ 繞 } x \text{ 軸旋轉一圈 } \begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = f(r) \end{cases}, r = a \cosh \frac{z}{a}$$

$$X(r, \theta) = [r \cos \theta, r \sin \theta, a \cosh^{-1} \frac{r}{a}], \text{ 則}$$

$$X_r = [\cos \theta, \sin \theta, \frac{a}{\sqrt{r^2 - a^2}}], X_\theta = [-r \sin \theta, r \cos \theta, 0]$$

$$X_{rr} = [0, 0, \frac{-ar}{(r^2 - a^2)^{\frac{3}{2}}}], X_{r\theta} = [-\sin \theta, \cos \theta, 0], X_{\theta\theta} = [-r \cos \theta, -r \sin \theta, 0]$$

$$E = X_r \cdot X_r = \frac{r^2}{r^2 - a^2}, F = 0, G = X_\theta \cdot X_\theta = r^2$$

$$X_r \times X_\theta = [\frac{-ar \cos \theta}{\sqrt{r^2 - a^2}}, \frac{ar \sin \theta}{\sqrt{r^2 - a^2}}, r]$$

$$N = \frac{X_r \times X_\theta}{|X_r \times X_\theta|} = [\frac{-a \cos \theta}{r}, \frac{a \sin \theta}{r}, \frac{\sqrt{r^2 - a^2}}{r}]$$

$$e = X_{rr} \cdot N = \frac{-a}{r^2 - a^2}, f = 0, g = X_{\theta\theta} \cdot N = a$$

$$H = \frac{Eg - 2Ff + Ge}{2(EG - F^2)} = \dots = 0$$

或者寫成

$$X(u, v) = (a \cosh v \cos u, a \cosh v \sin u, av)$$

$$0 < u < 2\pi, -\infty < v < \infty$$

$$\text{Then } E = G = a^2 \cosh^2 v, F = 0$$

$$\text{Check } X_{uu} + X_{vv} = 0$$