§ 流形上有三個微分運算

1. 外微分 d：針對 differential forms
2. 協變微分 $\nabla_X Y$ ：Y 在 X 方向的導數取切部。
3. 李導數 $L_X Y$ or $L_X \omega$

其中 covariant derivtive for vector fields

$$X=\sum_i X^i \frac{\partial}{\partial x^i}, Y=\sum_i Y^i \frac{\partial}{\partial x^i}$$ ，則 $\nabla_X Y=\sum_i (XY^i+\sum_{j,k}\Gamma^i_{jk}X^jY^k)\frac{\partial}{\partial x^i}$

寫成 $\nabla_\mu V^\nu=\partial_\mu V^\nu+\Gamma^\nu_{\mu\sigma}V^\sigma$

Covariant derivative of a 1-form ω

$\nabla_X \omega=\sum_i (X\omega_i-\sum_{j,k}\Gamma^k_{ji}X^j\omega_k)dx^i$ ，寫成 $\nabla_\mu\omega_\nu=\partial_\mu\omega_\nu-\Gamma^\lambda_{\mu\nu}\omega_\lambda$

例如 metric compatibility $\Leftrightarrow \nabla_\rho g_{\mu\nu}=0$

$L_X\omega=\lim_{t\to 0}\frac{1}{t}(\varphi_t^*\omega-\omega)=\frac{d}{dt}\left(\varphi_t^*\right)\big|_{t=0}$ ，then

1. $L_X(\omega_1\wedge\omega_2)=(L_X\omega_1)\wedge\omega_2+\omega_1\wedge(L_X\omega_2)$
2. $d(L_X\omega)=L_X d\omega$
3. $L_X\omega=\iota_X d\omega+d(\iota_X\omega)$
4. $L_X(\iota_Y\omega)=\iota_{L_XY}\omega+\iota_Y L_X\omega$
5. Let $\omega=\sum_i h_i dx^i, X=\sum_i \xi^i\frac{\partial}{\partial x^i}$ ，then $L_X\omega=\sum_j (Xh_j)dx^j+\sum_k h_k d\xi^k$

1. $$L_X(\omega_1\wedge\omega_2)=\frac{d}{dt}\varphi_t^*(\omega_1\wedge\omega_2)\big|_{t=0}=\frac{d}{dt}\left(\varphi_t^*\omega_1\right)\wedge\left(\varphi_t^*\omega_2\right)\big|_{t=0}$$
$$=\left(\frac{d}{dt}(\varphi_t^*\omega_1)\big|_{t=0}\right)\wedge\omega_2+\omega_1\wedge\left(\frac{d}{dt}(\varphi_t^*\omega_2)\big|_{t=0}\right)=(L_X\omega_1)\wedge\omega_2+\omega_1\wedge(L_X\omega_2)$$
2. …
3. …
4. …
5. (Let $\omega=\omega_i d\omega^i, X=X^j\frac{\partial}{\partial x^j}$ ，then $L_X\omega=(X^j\frac{\partial\omega_k}{\partial x^j}+\omega_j\frac{\partial x^j}{\partial x^k})dx^k$)

Cartan magic formula $L_X\omega = \iota_X d\omega + d(\iota_X\omega)$

右式前者是 interior product(interior derivatve)，後者是 exterior derivative

外微分後做內積 + 做內積後再外微分。

1. 例子 (in RG1102)

$X = F\frac{\partial}{\partial x} + G\frac{\partial}{\partial y} + H\frac{\partial}{\partial z}$ ，volume form $\omega = dv = dx \wedge dy \wedge dz$

$L_X dx = d(L_X x) = d(Xx) = dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz$

Then

$L_X dv = L_X(dx \wedge dy \wedge dz) = (L_X dx)\wedge dy \wedge dz + dx \wedge (L_X dy)\wedge dz + dx \wedge dy \wedge (L_X dz)$

$= (\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z})dv = (divX)dv$

右式 $\iota_X\omega = Fdy \wedge dz - Gdx \wedge dz + Hdx \wedge dy$

$d(\iota_X\omega) = dF \wedge dy \wedge dz - dG \wedge dx \wedge dz + dH \wedge dx \wedge dy$

其中 $dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz$ ，…。 $d\omega = d^2 v = 0$

所以 $d(\iota_X\omega) = (\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z})dx \wedge dy \wedge dz$

所謂的 interior product 如何運算。

$\iota_X(dx_1 \wedge dx_2 \ldots \wedge dx_n) = \sum_{r=1}^{n}(-1)^{r-1} f_r dx_1 \wedge \ldots \wedge \hat{dx_r} \wedge \ldots \wedge dx_n$

這裡 $\hat{dx_r}$ 表示把 dx_r 省略

2. 證明

[Elementary Proof of the Cartan Formula]

3. 應用

X is a Killing vector field $\Leftrightarrow L_X g = 0$

$(L_X g)(Z,W) = g(\nabla_Z X, W) + g(Z, \nabla_W X)$

X is a harmonic vector field $\Leftrightarrow dX^b = 0$ and div X=0