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Examples and Structure of CMC Surfaces in Some Riemannian and Lorentzian Homogeneous Spaces

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1. Introduction

Abresch and Rosenberg have proved the existence of a quadratic differential for an immersed surface in $\mathbb{M}^2(\kappa) \times \mathbb{R}$ that is holomorphic when the surface has constant mean curvature (CMC). Here, $\mathbb{M}^2(\kappa)$ denotes the 2-dimensional simply connected space form with constant curvature κ. This differential Q plays the role of the usual Hopf differential in the theory of CMC surfaces immersed in space forms. Thus, Abresch and Rosenberg were able to prove the following theorem.

Theorem [1, Thm. 2]. *Any immersed CMC sphere $S^2 \looparrowright \mathbb{M}^2(\kappa) \times \mathbb{R}$ in a product space is actually one of the embedded rotationally invariant CMC spheres $S^2_H \subset \mathbb{M}^2(\kappa) \times \mathbb{R}$.*

The rotationally invariant spheres referred to here were constructed independently by W.-Y. Hsiang and W.-T. Hsiang in [9] and by Pedrosa and Ritoré in [14] and [15]. The quoted theorem proves affirmatively a conjecture stated by Hsiang and Hsiang in their paper [9]. More importantly, it indicates that some tools often used for surface theory in space forms could be redesigned for more general 3-dimensional homogeneous spaces—the more natural ones after space forms being $\mathbb{M}^2(k) \times \mathbb{R}$. The price to be paid in abandoning space forms is that the technical difficulties are more involved. The method in [1] is to study closely the revolution surfaces in $\mathbb{M}^2(\kappa) \times \mathbb{R}$ in order to guess the suitable differential.

Our idea is to relate the Q differential on a surface Σ immersed in $\mathbb{M}^2(\kappa) \times \mathbb{R}$ with the usual Hopf differential after embedding $\mathbb{M}^2(\kappa) \times \mathbb{R}$ in some Euclidean space $\mathbb{E}^4$. We prove that Q may be written as a linear combination of the Hopf differentials Ψ^1 and Ψ^2 associated to two normal directions spanning the normal bundle of Σ in $\mathbb{E}^4$. This is true also when the product $\mathbb{M}^2(\kappa) \times \mathbb{R}$ carries a Lorentzian metric. More precisely, defining r as $r^2 = \epsilon/\kappa$ for $\epsilon = \operatorname{sgn} \kappa$ allows us to state the following result.

Theorem (Theorem 4). *The quadratic differential $Q = 2H\Psi^1 - \varepsilon \frac{\epsilon}{r} \Psi^2$ is holomorphic on $\Sigma \looparrowright \mathbb{M}^2(\kappa) \times \mathbb{R}$ if the mean curvature H of Σ is constant.*

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Our aim here is to explore the geometrical consequences of this alternative presentation of Q. The paper proceeds as follows. Sections 2 and 3 are concerned with describing rotationally invariant examples of CMC surfaces on both Riemannian and Lorentzian products. In particular, the explicit formulas for CMC revolution discs and spheres with $Q = 0$ presented in [1] (see also [17]) are reobtained in Section 2 by elementary methods. In Section 4 we present the proof of the Theorem 4 and a variant of the classical Theorem of Joachimstahl, which gives a characterization of CMC rotationally invariant discs and spheres in the same spirit of the result by Abresch and Rosenberg mentioned previously (see Theorem 5).

We also prove in Section 5 the following result about free boundary CMC surfaces, which is based on Nitsche's well-known work on the partitioning problem.

Theorem (Theorem 6). *Let Σ be a surface immersed in $\mathbb{M}^2(\kappa) \times \mathbb{R}$ whose boundary is contained in some horizontal plane $\mathbb{P}_a$. Suppose that Σ has constant mean curvature and that its angle with $\mathbb{P}_a$ is constant along its boundary. If $\varepsilon = 1$ and Σ is disc-type, then Σ is a spherical cap. If $\varepsilon = -1$, then Σ is a hyperbolic cap.*

The variational meaning of the conditions on Σ can be seen in Section 5. We end this section with a characterization of stable CMC discs with circular boundary on $\mathbb{M}^2(\kappa) \times \mathbb{R}$ that generalizes a nice result of Alías, López, and Palmer (see [3]). Finally, in Section 6 we obtain estimates of some geometrical data of CMC surfaces with boundary lying on vertical planes in $\mathbb{M}^2(\kappa) \times \mathbb{R}$.

In a forthcoming paper (see [10]), one of the authors elaborates versions of the results contained here for constant mean curvature hypersurfaces in some homogeneous spaces and warped products. There, a suitable treatment of Minkowski formulas gives some hints about stability problems and the existence of CMC Killing graphs.

2. Rotationally Invariant CMC Surfaces

Let $\mathbb{M}^2(\kappa)$ be a 2-dimensional simply connected surface endowed with a Riemannian complete metric $d\sigma^2$ with constant sectional curvature κ. We fix the metric $\varepsilon dt^2 + d\sigma^2$ ($\varepsilon = \pm 1$) on the product $\mathbb{M}^2(\kappa) \times \mathbb{R}$. This metric is Lorentzian if $\varepsilon = -1$ and Riemannian if $\varepsilon = 1$.

A tangent vector v to $\mathbb{M}^2(\kappa) \times \mathbb{R}$ is projected onto a horizontal component v^h and a vertical component v^t, which are tangent to the $T\mathbb{M}^2(\kappa)$ and $T\mathbb{R}$ factors, respectively. We denote by $\langle \cdot, \cdot \rangle$ and D, respectively, the metric and covariant derivative in $\mathbb{M}^2(\kappa) \times \mathbb{R}$. The curvature tensor in $\mathbb{M}^2(\kappa) \times \mathbb{R}$ is denoted by $\bar{R}$.

Let (ρ, θ) be polar coordinates centered at some point p_0 in $\mathbb{M}^2(\kappa)$, and let (ρ, θ, t) be the corresponding cylindrical coordinates in $\mathbb{M}^2(\kappa) \times \mathbb{R}$. Then fix a curve $s \mapsto (\rho(s), 0, t(s))$ in the plane $\theta = 0$. If we rotate this curve around the t-axis, we obtain a rotationally invariant surface (for short, a *revolution surface*) Σ in $\mathbb{M}^2(\kappa) \times \mathbb{R}$ whose axis is $\{p_0\} \times \mathbb{R}$. In other words, this surface has a parameterization X, in terms of the cylindrical coordinates just defined, of the following form:

$$X(s,\theta) = (\rho(s), \theta, t(s)). \tag{1}$$

In Lorentzian products, we will consider only *spacelike* revolution surfaces—that is, surfaces for which the metric induced on them is a Riemannian metric.

One may easily verify that X has constant mean curvature H if and only if the following equation is satisfied:

$$2HW^3 = (\ddot{\rho}\dot{t} - \ddot{t}\dot{\rho})\,\mathrm{sn}_\kappa^3(\rho) - \dot{t}\dot{\rho}^2\,\mathrm{sn}_\kappa^2(\rho)\,\mathrm{cs}_\kappa(\rho), \tag{2}$$

where $W^2 = \mathrm{sn}_\kappa^2(\rho)(\dot{\rho}^2 + \varepsilon\dot{t}^2)$ and derivatives are taken with respect to s. We suppose momentarily that the profile curve $(\rho(s), 0, t(s))$ is given as a graph $t = t(\rho)$. Thus, considering $\rho = s$ above, one can easily verify that the expression

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{\dot{t}\,\mathrm{sn}_\kappa^2(\rho)}{W}\right) = -2H\,\mathrm{sn}_\kappa(\rho)$$

is equivalent to equation (2). This means that

$$\frac{\mathrm{d}t}{\mathrm{d}\rho}\frac{\mathrm{sn}_\kappa^2(\rho)}{W} = I - 2H\int \mathrm{sn}_\kappa(\rho)\,\mathrm{d}\rho \tag{3}$$

is a first integral to the mean curvature equation (2) associated to translations on the t-axis.

One may then prove that, for a suitable choice of parameters, the differential Q (see Section 4) has constant coefficient ψ for any revolution surface with constant mean curvature H:

$$\psi = -\frac{1}{2\kappa}(\kappa^2 I^2 - 4H^2)$$

for $\kappa \neq 0$. From the same calculations, we assure that the Hopf differential has constant coefficient $\psi^1 = I$ for $\kappa = 0$. Thus, the CMC rotational examples for $\kappa \neq 0$ have $Q = 0$ if and only if $I = \pm 2H/\kappa$. We therefore replace $I = \pm 2H/\kappa$ in (3). Since $W^2 = \mathrm{sn}_\kappa^2(\rho)\big(1 + \varepsilon\big(\frac{\mathrm{d}t}{\mathrm{d}\rho}\big)^2\big)$, it follows that

$$\frac{\frac{\mathrm{d}t}{\mathrm{d}\rho}}{\sqrt{1+\varepsilon\big(\frac{\mathrm{d}t}{\mathrm{d}\rho}\big)^2}}\,\mathrm{sn}_\kappa(\rho) = -\frac{2H}{\kappa}\left(\pm 1 + \kappa\int \mathrm{sn}_\kappa(\rho)\,\mathrm{d}\rho\right) = -\frac{2H}{\kappa}(\pm 1 - \mathrm{cs}_\kappa(\rho)).$$

As a result, for $I = -2H/\kappa$ we have

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\right)^2 + \varepsilon = \frac{\kappa}{4H^2}\frac{1+\mathrm{cs}_\kappa(\rho)}{1-\mathrm{cs}_\kappa(\rho)}.$$

However,

$$\frac{1+\mathrm{cs}_\kappa(\rho)}{1-\mathrm{cs}_\kappa(\rho)} = \frac{1}{\kappa}\,\mathrm{ct}_\kappa^2\left(\frac{\rho}{2}\right) = \frac{1}{\kappa}\frac{1}{r^2}.$$

Here $\mathrm{ct}_\kappa(\rho) = \dot{\mathrm{sn}}_\kappa(\rho)/\mathrm{sn}_\kappa(\rho)$ is the geodesic curvature of the geodesic circle centered at p_0 with radius ρ in $\mathbb{M}^2(\kappa)$, and r is the Euclidean radial distance measured from p_0 on the Euclidean model for $\mathbb{M}^2(\kappa)$. Thus the resulting equation is

$$\frac{2}{1+\kappa r^2}\frac{2Hr\,\mathrm{d}r}{\sqrt{1-4H^2\varepsilon r^2}} = \mathrm{d}t.$$

We change variables by defining (for $\kappa < 0$) $v = \varepsilon u - (\varepsilon + \kappa/4H^2)$ and $v = (\varepsilon + \kappa/4H^2) - \varepsilon u$ (for $\kappa > 0$), where $u = 1 + \kappa r^2$. Next, we put $w = \sqrt{v}$. Hence $\mathrm{d}v = 2w\mathrm{d}w$, and the final form of the equation is

$$\frac{2\mathrm{d}w}{w^2 + (\varepsilon + \kappa/4H^2)} = -\sqrt{-\kappa}\,\mathrm{d}t, \quad \kappa < 0,$$
$$\frac{2\mathrm{d}w}{w^2 - (\varepsilon + \kappa/4H^2)} = \sqrt{\kappa}\,\mathrm{d}t, \quad \kappa > 0.$$

We suppose that $4H^2\varepsilon + \kappa > 0$. Then

$$(4H^2\varepsilon + \kappa)\,\mathrm{sn}^2_\kappa(\rho/2) + 4H^2\varepsilon\,\mathrm{sn}^2_{-\kappa}(ct/2) = 1 \quad (\kappa < 0), \tag{4}$$

where $c = \sqrt{\varepsilon + \kappa/4H^2}$ and $\varepsilon = 1$. The same formula holds for $\kappa > 0$ and $\varepsilon = 1$. For $\kappa > 0$ and $\varepsilon = -1$,

$$4H^2\varepsilon\kappa\,\mathrm{sn}^2_{-\kappa}(ct/2) = -(4H^2\varepsilon + \kappa)\,\mathrm{cs}^2_\kappa(\rho/2). \tag{5}$$

We now treat the case $\varepsilon + \kappa/4H^2 < 0$, denoting $c^2 = -(\varepsilon + \kappa/4H^2)$. Thus, for $\kappa > 0$ and $\varepsilon = -1$, the solution is

$$(4H^2\varepsilon + \kappa)\,\mathrm{sn}^2_\kappa(\rho/2) - 4H^2\varepsilon\,\mathrm{sn}^2_\kappa(ct/2) = 1. \tag{6}$$

The same formula holds for $\kappa < 0$ and $\varepsilon = -1$ when $|w| < c$. For $\varepsilon = 1$, we necessarily have $\kappa < 0$ and $|w| > c$. Thus

$$4H^2\varepsilon\kappa\,\mathrm{sn}^2_\kappa(ct/2) = (4H^2\varepsilon + \kappa)\,\mathrm{cs}^2_\kappa(\rho/2). \tag{7}$$

Finally, for $\varepsilon + \kappa/4H^2 = 0$ one obtains

$$t^2 = \varepsilon(4/\kappa)\,\mathrm{cs}^2_\kappa(\rho/2). \tag{8}$$

Next, we consider $I = 2H/\kappa$. First suppose $c^2 = \varepsilon + \kappa/4H^2 > 0$. In this case there are no examples with $\kappa < 0$. For $\kappa > 0$ and $\varepsilon = 1$,

$$(4H^2\varepsilon + \kappa)\kappa\,\mathrm{sn}^2_\kappa(\rho/2) = 4H^2\varepsilon\,\mathrm{cs}^2_{-\kappa}\big(\sqrt{\varepsilon + \kappa/4H^2}\,t/2\big). \tag{9}$$

For $\kappa > 0$ and $\varepsilon = -1$,

$$(4H^2\varepsilon + \kappa)\,\mathrm{sn}^2_\kappa(\rho/2) = -4H^2\varepsilon\,\mathrm{sn}^2_{-\kappa}\big(\sqrt{\varepsilon + \kappa/4H^2}\,t/2\big). \tag{10}$$

Now we consider the case $-c^2 = \varepsilon + \kappa/4H^2 < 0$. For $\kappa < 0$ and $\varepsilon = 1$, we have

$$(4H^2\varepsilon + \kappa)\kappa\,\mathrm{sn}^2_\kappa(\rho/2) = 4H^2\varepsilon\,\mathrm{cs}^2_\kappa\big(\sqrt{-(\varepsilon + \kappa/4H^2)}\,t/2\big). \tag{11}$$

The same expression holds for $\kappa > 0$ and $\varepsilon = -1$. For $\kappa < 0$ and $\varepsilon = -1$,

$$(4H^2\varepsilon + \kappa)\,\mathrm{sn}^2_\kappa(\rho/2) = 4H^2\varepsilon\,\mathrm{sn}^2_\kappa\big(\sqrt{-(\varepsilon + \kappa/4H^2)}\,t/2\big). \tag{12}$$

THEOREM 1. *The revolution surfaces with constant mean curvature H and $Q = 0$ on $\mathbb{M}^2(\kappa) \times \mathbb{R}$ correspond to the values $I = \pm 2H/\kappa$. These surfaces are described by* (4)–(12).

For $\varepsilon = 1$, the preceding formulas were already obtained in [1] by other integration methods.

3. Rotationally Invariant CMC Discs on Lorentzian Products

3.1. Qualitative Description

In this section we consider only spacelike revolution surfaces in Lorentzian products $\mathbb{M}^2(\kappa) \times \mathbb{R}$ with $\kappa \leq 0$. We assume that the parameter s on (1) is the arc length of the profile curve, so $\dot{\rho}^2 - \dot{t}^2 = 1$. We denote by φ the hyperbolic angle with the horizontal axis ∂_ρ. Then Σ has constant mean curvature H if and only if $(\rho(s), t(s), \varphi(s))$ is a solution to the following ordinary differential equations system:

$$\begin{cases} \dot{\rho} = \cosh\varphi, \\ \dot{t} = \sinh\varphi, \\ \dot{\varphi} = -2H - \sinh\varphi\,\mathrm{ct}_\kappa(\rho). \end{cases} \tag{13}$$

The *flux* I' through a horizontal plane $\mathbb{P}_t = \mathbb{M}^2(\kappa) \times \{t\}$ is, up to a constant, given by the expression for I in terms of s:

$$I' = I + \frac{2H}{\kappa} = \sinh\varphi\,\mathrm{sn}_\kappa(\rho) + 2H \int_0^\rho \mathrm{sn}_\kappa(\tau)\,\mathrm{d}\tau. \tag{14}$$

Integrating the last term in (14) yields

$$I' = \sinh\varphi\,\mathrm{sn}_\kappa(\rho) + 4H\,\mathrm{sn}_\kappa^2(\rho/2). \tag{15}$$

The solutions to (13) for which $Q = 0$ vanishes are those with $I = \pm 2H/\kappa$ or $I' = 0, 4H/\kappa$. We later give a qualitative description of these solutions.

Denoting $u = \sinh\varphi$, we conclude that the system (13) is equivalent to

$$\begin{cases} \dfrac{\mathrm{d}t}{\mathrm{d}\rho} = \dfrac{u}{\sqrt{1+u^2}}, \\ \dfrac{\mathrm{d}u}{\mathrm{d}\rho} = -2H - u\,\mathrm{ct}_\kappa(\rho). \end{cases} \tag{16}$$

It is clear that solutions of (16) are defined on the whole real line and that the profile curve may be written as a graph over the ρ-axis. Now we begin to describe the maximal solutions—that is, solutions for $H = 0$. If we consider a fixed value for I' then the condition $H = 0$ implies that

$$I' = \sinh\varphi\,\mathrm{sn}_\kappa(\rho). \tag{17}$$

Therefore, the horizontal planes are the unique maximal revolution surfaces with $I' = 0$. In fact, if we put $I' = 0$ at (17) then $\sinh\varphi = 0$ for $\rho > 0$. Thus $\dot{t} = 0$ and we conclude that the solution is a horizontal plane. Hence, we may assume $I \neq 0$. In this case, since $\mathrm{sn}_\kappa(\rho) \to 0$ if $\rho \to 0$, it follows that $\sinh\varphi \to \infty$ if $\rho \to 0$. Hence Σ has a singularity and asymptotes the light cone at p_0 (the light cone corresponds to $\varphi = \infty$). Moreover, $\sinh\varphi \to 0$ if $\rho \to \infty$ since $\kappa \leq 0$. This means that these maximal surfaces asymptote a horizontal plane for $\rho \to \infty$; that is, these surfaces have planar ends.

Consider now $H \neq 0$. In this case the solutions are regular if and only if $\varphi \to 0$ as $\rho \to 0$, which implies that $\sinh\varphi \to 0$ as $\rho \to 0$. Thus necessarily $I' = 0$, as we could see by taking the limit $\rho \to 0$ in (15). Hence, for systems (13) and (16), all examples of solutions that orthogonally touch the revolution axis have $I' = 0$. Reciprocally, if we put $I' = 0$ in (15) then

$$0 = \sinh\varphi\, \mathrm{sn}_\kappa(\rho) + 4H\, \mathrm{sn}_\kappa^2(\rho/2).$$

Dividing this expression by $2\,\mathrm{sn}_\kappa^2(\rho/2)$ yields

$$\sinh\varphi\, \mathrm{ct}_\kappa(\rho/2) = -2H. \tag{18}$$

One easily verifies that $\sinh\varphi \to 0$ if $\rho \to 0$, so all solutions for (16) with $I' = 0$ reach the revolution axis orthogonally as noted previously. Hence these solutions correspond to initial conditions $t(0) = t_0$, $\rho(0) = 0$, and $\varphi(0) = 0$ for system (13). Now

$$\mathrm{ct}_\kappa(\rho) = \frac{1}{2}\left(-\frac{2H}{\sinh\varphi} + \kappa\,\frac{\sinh\varphi}{2H}\right) = \frac{-4H^2 + \kappa\sinh^2\varphi}{4H\sinh\varphi}.$$

Substituting this into the third equation in (13), we obtain

$$\frac{\mathrm{d}\varphi}{\mathrm{d}s} = \frac{1}{4H}(-4H^2 - \kappa\sinh^2\varphi). \tag{19}$$

Observe that $\dot\varphi = -H$ is the corresponding equation for the case $\kappa = 0$—that is, for hyperbolic spaces in $\mathbb{L}^3$. This could be obtained as a limiting case if we take $\kappa \to 0$. For $\kappa < 0$, the range for the angle φ is $0 \le \varphi < \varphi_\infty = \operatorname{arcsinh}\bigl(2|H|/\sqrt{-\kappa}\,\bigr)$. The surface necessarily asymptotes a spacelike cone with angle φ_∞. Indeed, equation (19) is equivalent to

$$\frac{1}{4H}\int_0^{\varphi_\infty} \frac{\mathrm{d}\varphi}{-4H^2 - \kappa\sinh^2\varphi} = \int_0^\infty \mathrm{d}s = \infty.$$

Finally, we study the case when $\varphi \to \varphi_0$ as $\rho \to 0$ for some positive value of φ_0. This means that the solution asymptotes a spacelike cone at p_0. In this case, $\sinh\varphi \to \sinh\varphi_0 < \infty$ as $\rho \to 0$. Thus, taking the limit $\rho \to 0$ in (15) yields $I' = 0$. So, as we have already seen, necessarily $\varphi_0 = 0$. This contradiction implies that there are no examples with $\varphi_0 > 0$.

It remains to examine the case $\varphi \to \infty$ as $\rho \to 0$. In this case, the solution asymptotes the light cone at p_0. For any nonzero value of I', after dividing (15) by $\mathrm{sn}_\kappa^2(\rho/2)$ and taking limit for $\rho \to \infty$, we obtain that $\sinh\varphi \to 2|H|/\sqrt{-\kappa}$. Moreover, the angle φ is always decreasing in the range $\bigl(2|H|/\sqrt{-\kappa}, \infty\bigr)$ as ρ increases in $(0, \infty)$. For example, consider the values $\kappa < 0$ and $I' = 4H/\kappa$. Replacing this value for I' in (14) yields

$$0 = \sinh\varphi\, \mathrm{sn}_\kappa(\rho) + 4H(\mathrm{sn}_\kappa^2(\rho/2) - 1/\kappa),$$

so we conclude that

$$\kappa\sinh\varphi = 2H\, \mathrm{ct}_\kappa(\rho/2). \tag{20}$$

Thus the solution satisfies $\sinh\varphi \to \infty$ if $\rho \to 0$. This means that Σ asymptotes the light cone at the point p_0. Moreover, we have that $\sinh\varphi \to 2|H|/\sqrt{-\kappa}$ if $\rho \to \infty$. Substituting (20) into the third equation in (13), we obtain

$$\mathrm{ct}_\kappa(\rho) = \frac{1}{2}\left(\kappa\,\frac{\sinh\varphi}{2H} - \frac{2H}{\sinh\varphi}\right) = \frac{-4H^2 + \kappa\sinh^2\varphi}{4H\sinh\varphi}$$

and

$$\frac{\mathrm{d}\varphi}{\mathrm{d}s} = \frac{1}{4H}(-4H^2 - \kappa\sinh^2\varphi).$$

Because φ satisfies $\sinh\varphi > \sinh\varphi_\infty = 2|H|/\sqrt{-\kappa}$, we conclude that $\dot\varphi < 0$ for all s. Hence the angle decreases from ∞ at $\rho \to 0$ to its infimum value φ_∞ as $\rho \to \infty$.

We summarize these facts in the following theorem.

THEOREM 2. *Let Σ be a rotationally invariant surface with constant mean curvature H in the Lorentzian product $\mathbb{M}^2(\kappa)\times\mathbb{R}$ with $\kappa \le 0$. If $H = 0$, then either Σ is a horizontal plane $\mathbb{P}_t = \mathbb{M}^2(\kappa)\times\{t\}$ or Σ asymptotes a light cone with vertex at some point p_0 of the rotation axis. In the latter case, Σ has a singularity at p_0 and has horizontal planar ends. We refer to these singular surfaces as* Lorentzian catenoids.

If $H \neq 0$, then either Σ is a complete disc-type surface orthogonally meeting the rotation axis or Σ asymptotes a light cone with vertex p_0 at the rotation axis. In the first case, the angle between the surface and the horizontal planes asymptotes $2|H|/\sqrt{-\kappa}$ as the surface approaches the asymptotic boundary $\partial_\infty\mathbb{M}^2(\kappa)\times\mathbb{R}$. In the last case, the surface is singular at p_0 and asymptotes a spacelike cone with vertex at p_0 and slope φ_∞, where $\sinh\varphi_\infty = 2|H|/\sqrt{-\kappa}$.

3.2. Uniqueness of Annular CMC Surfaces

We fix $\varepsilon = -1$ and $\kappa \le 0$ throughout this section. We shall present a version of a theorem proved by López (see [11, Thm. 1.2]) about uniqueness of annular CMC in Minkowski space $\mathbb{L}^3$.

Let Σ_1 be a connected CMC spacelike surface in $\mathbb{M}^2(\kappa)\times\mathbb{R}$ whose boundary is a geodesic circle Γ in some plane $\mathbb{P}_a$. We suppose that Σ_1 is a graph over $\mathbb{P}_a - \Omega$, where Ω is the domain bounded by Γ on $\mathbb{P}_a$. We further suppose that, when Σ_1 approaches $\partial_\infty\mathbb{M}^2(\kappa)\times\mathbb{R}$, the angle of Σ_1 with respect to the planes $\mathbb{P}_t$ asymptotes a value φ^1_∞ such that $\sinh(\varphi^1_\infty) \ge 2|H|/\sqrt{-\kappa}$. Now consider Σ_2 a revolution surface with the same mean curvature, boundary, and flux as Σ_1; that this is possible we infer from the description in Theorem 2. By the same theorem we know that the asymptotic angle for Σ_2 is $\varphi^2_\infty = \operatorname{arcsinh}\big(2|H|/\sqrt{-\kappa}\,\big)$.

Suppose that $\Sigma_1 \neq \Sigma_2$, and move Σ_1 upward until there is no contact with Σ_2. This is possible because the asymptotic angle of Σ_1 is greater than or equal to the asymptotic angle of Σ_2. The maximum principle allows us to obtain a contradiction. For that, it suffices to mimic the proof of Theorem 1.2 in [11]. Thus, we conclude that $\Sigma_1 = \Sigma_2$ and the proof of the theorem.

THEOREM 3. *Let Σ be a spacelike CMC surface on $\mathbb{M}^2(\kappa)\times\mathbb{R}$, $\kappa\le 0$, whose boundary is a geodesic circle on a horizontal plane $\mathbb{P}_a$. We suppose that Σ is a graph over the domain in $\mathbb{P}_a$ outside the disc bounded by $\partial\Sigma$. We further suppose that the angle between Σ and the horizontal planes asymptotes φ_∞ with $\varphi_\infty \ge$ $\operatorname{arcsinh}\big(2|H|/\sqrt{-\kappa}\big)$. Then Σ is contained on a revolution surface whose axis passes through the center of $\partial\Sigma$ on $\mathbb{P}_a$.*

A similar reasoning shows, under the same hypothesis on the asymptotic angle, that an entire spacelike surface with an isolated singularity and constant mean curvature is a singular revolution surface (see [11, Thm. 1.3]).

4. Hopf Differentials in Some Product Spaces

Let Σ be a Riemann surface and let $X\colon \Sigma\to\mathbb{M}^2(\kappa)\times\mathbb{R}$ be an isometric immersion. If $\kappa\ge 0$, we may consider Σ as immersed in $\mathbb{R}^4=\mathbb{R}^3\times\mathbb{R}$. If $\kappa<0$, we immerse Σ in $\mathbb{L}^3\times\mathbb{R}$. In fact, we may write $X=(p,t)$, with $t\in\mathbb{R}$ and $p\in\mathbb{M}^2(\kappa)\subset\mathbb{R}^3$ for $\kappa\ge 0$ and $p\in\mathbb{M}^2(\kappa)\subset\mathbb{L}^3$ for $\kappa<0$. By writing $\mathbb{M}^2(\kappa)\times\mathbb{R}\subset\mathbb{E}^4$ we mean all these possibilities. The metric and covariant derivative in $\mathbb{E}^4$ are also denoted by $\langle\cdot,\cdot\rangle$ and D respectively. We denote by ϵ the sign of κ. Recall that $\varepsilon=1$ for Riemannian products and $\varepsilon=-1$ for Lorentzian ones.

Let (u,v) be local coordinates in Σ for which $X(u,v)$ is a conformal immersion inducing the metric $e^{2\omega}(\mathrm{d}u^2+\mathrm{d}v^2)$ in Σ. We denote by ∂_u,∂_v the coordinate vectors and let $e_1=e^{-\omega}\partial_u$ and $e_2=e^{-\omega}\partial_v$ constitute the associated local orthonormal frame tangent to Σ. The unit normal directions to Σ in $\mathbb{E}^4$ are denoted by $n_1, n_2=p/r$, where $r=(\epsilon\langle p,p\rangle)^{1/2}$. We denote by h^k_{ij} the components of h^k, the second fundamental form of Σ with respect to n_k, $k=1,2$. Then

$$h^k_{ij}=\langle D_{e_i}e_j,n_k\rangle.$$

It is clear that the h^1_{ij} are the components of the second fundamental form of the immersion $\Sigma\hookrightarrow\mathbb{M}^2(\kappa)\times\mathbb{R}$. The components of h^2 are

$$\begin{aligned} h^2_{ij} &= \langle D_{e_i}e_j,n_2\rangle=\Big\langle D_{e_i^h}e_j^h,\frac{p}{r}\Big\rangle=-\frac{1}{r}\langle e_i^h,e_j^h\rangle=\frac{1}{r}(\varepsilon\langle e_i^t,e_j^t\rangle-\delta_{ij})\\ &=\frac{1}{r}(\varepsilon\langle e_i,\partial_t\rangle\langle e_j,\partial_t\rangle-\delta_{ij})=\frac{\varepsilon}{r}\langle e_i,\partial_t\rangle\langle e_j,\partial_t\rangle-\frac{1}{r}\delta_{ij}. \end{aligned}$$

We remark that $\kappa=\epsilon/r^2$. The components of h^1 and h^2 in the frame ∂_u,∂_v are respectively

$$e=h^1(\partial_u,\partial_u)=e^{2\omega}h^1_{11},\quad f=h^1(\partial_u,\partial_v)=e^{2\omega}h^1_{12},\quad g=h^1(\partial_v,\partial_v)=e^{2\omega}h^1_{22}$$

and

$$\tilde e=h^2(\partial_u,\partial_u)=e^{2\omega}h^2_{11},\quad \tilde f=h^2(\partial_u,\partial_v)=e^{2\omega}h^2_{12},\quad \tilde g=h^2(\partial_v,\partial_v)=e^{2\omega}h^2_{22}.$$

The Hopf differential associated to h^k is defined by $\Psi^k = \psi^k \mathrm{d}z^2$, where $z = u + iv$ and the coefficients ψ^1, ψ^2 are

$$\psi^1 = \frac{1}{2}(e-g) - if, \qquad \psi^2 = \frac{1}{2}(\tilde{e}-\tilde{g}) - i\tilde{f}.$$

The mean curvature of X is by definition $H = (h^1_{11} + h^1_{22})/2$. Differentiating the real part of ψ^1, we obtain

$$\begin{aligned}
&\partial_u\left(\frac{e-g}{2}\right)\\
&\quad= \partial_u\left(\frac{e+g}{2} - g\right) = \partial_u(e^{2\omega}H) - \partial_u g = \partial_u(e^{2\omega}H) - \partial_u(h^1(\partial_v,\partial_v))\\
&\quad= \partial_u(e^{2\omega}H) - (D_{\partial_u}h^1(\partial_v,\partial_v) + 2h^1(D_{\partial_u}\partial_v,\partial_v))\\
&\quad= \partial_u(e^{2\omega}H) - (D_{\partial_v}h^1(\partial_u,\partial_v) + \langle \bar{R}(\partial_u,\partial_v)n_1,\partial_v\rangle + 2h^1(D_{\partial_u}\partial_v,\partial_v))\\
&\quad= \partial_u(e^{2\omega}H) - (\partial_v(h^1(\partial_u,\partial_v)) - h^1(D_{\partial_v}\partial_u,\partial_v) - h^1(\partial_u, D_{\partial_v}\partial_v)\\
&\qquad\qquad + \langle \bar{R}(\partial_u,\partial_v)n_1,\partial_v\rangle + 2h^1(D_{\partial_u}\partial_v,\partial_v))\\
&\quad= \partial_u(e^{2\omega}H) - (\partial_v f + \Gamma^1_{12} f + \Gamma^2_{12} g - \Gamma^1_{22} e - \Gamma^2_{22} f + \langle \bar{R}(\partial_u,\partial_v)n_1,\partial_v\rangle)\\
&\quad= \partial_u(e^{2\omega}H) - (\partial_v f + f\partial_v\omega + g\partial_u\omega + e\partial_u\omega - f\partial_v\omega + \langle \bar{R}(\partial_u,\partial_v)n_1,\partial_v\rangle)\\
&\quad= \partial_u(e^{2\omega}H) - (\partial_v f + (e+g)\partial_u\omega + \langle \bar{R}(\partial_u,\partial_v)n_1,\partial_v\rangle)\\
&\quad= \partial_u(e^{2\omega}H) - 2e^{2\omega}H\partial_u\omega - \partial_v f - \langle \bar{R}(\partial_u,\partial_v)n_1,\partial_v\rangle\\
&\quad= -\partial_v f + e^{2\omega}\partial_u H - \langle \bar{R}(\partial_u,\partial_v)n_1,\partial_v\rangle.
\end{aligned}$$

Similar calculations yield

$$\partial_v\left(\frac{e-g}{2}\right) = \partial_u f - e^{2\omega}\partial_v H + \langle \bar{R}(\partial_v,\partial_u)n_1,\partial_u\rangle,$$

where we have used the Codazzi equation

$$D_{\partial_u}h^1(\partial_v,\partial_v) = D_{\partial_v}h^1(\partial_u,\partial_v) + \langle \bar{R}(\partial_u,\partial_v)n_1,\partial_v\rangle$$

and the following expressions for the Christoffel symbols Γ^k_{ij} for the metric $e^{2\omega}\delta_{ij}$ in Σ:

$$\Gamma^1_{11} = -\Gamma^1_{22} = \Gamma^2_{12} = \partial_u\omega, \qquad \Gamma^2_{22} = -\Gamma^2_{11} = \Gamma^1_{12} = \partial_v\omega.$$

An easy calculation yields the components of the curvature tensor:

$$\langle \bar{R}(\partial_u,\partial_v)n_1,\partial v\rangle = \kappa e^{2\omega}\langle \partial^h_u, n^h_1\rangle, \qquad \langle \bar{R}(\partial_v,\partial_u)n_1,\partial u\rangle = \kappa e^{2\omega}\langle \partial^h_v, n^h_1\rangle.$$

We thus obtain the following pair of equations:

$$\partial_u \Re\psi^1 = \partial_v \Im\psi^1 - \kappa e^{2\omega}\langle \partial^h_u, n^h_1\rangle + e^{2\omega}\partial_u H, \tag{21}$$

$$\partial_v \Re\psi^1 = -\partial_u \Im\psi^1 + \kappa e^{2\omega}\langle \partial^h_v, n^h_1\rangle - e^{2\omega}\partial_v H. \tag{22}$$

One may also calculate

$$\begin{aligned}
\partial_u \Re\psi^2 &= \frac{\varepsilon}{2r}\partial_u(\langle\partial_u,\partial_t\rangle^2 - \langle\partial_v,\partial_t\rangle^2)\\
&= \frac{\varepsilon}{r}(\langle\partial_u,\partial_t\rangle\langle D_{\partial_u}\partial_u,\partial_t\rangle - \langle\partial_v,\partial_t\rangle\langle D_{\partial_u}\partial_v,\partial_t\rangle)\\
&= \frac{\varepsilon}{r}(\langle\partial_u,\partial_t\rangle\langle D_{\partial_u}\partial_u,\partial_t\rangle - \langle\partial_v,\partial_t\rangle\langle D_{\partial_v}\partial_u,\partial_t\rangle)\\
&= \frac{\varepsilon}{r}(\langle\partial_u,\partial_t\rangle\langle D_{\partial_u}\partial_u,\partial_t\rangle - \partial_v(\langle\partial_v,\partial_t\rangle\langle\partial_u,\partial_t\rangle) + \langle D_{\partial_v}\partial_v,\partial_t\rangle\langle\partial_u,\partial_t\rangle)\\
&= \frac{\varepsilon}{r}\langle\partial_u,\partial_t\rangle\langle D_{\partial_u}\partial_u + D_{\partial_v}\partial_v,\partial_t\rangle - \frac{\varepsilon}{r}\partial_v(\langle\partial_u,\partial_t\rangle\langle\partial_v,\partial_t\rangle)\\
&= \frac{1}{r}\langle\partial_u,\partial_t\rangle e^{2\omega}\Delta t - \frac{\varepsilon}{r}\partial_v(\langle\partial_u,\partial_t\rangle\langle\partial_v,\partial_t\rangle)\\
&= 2H\frac{1}{r}e^{2\omega}\langle\partial_u,\partial_t\rangle\langle n_1,\partial_t\rangle - \frac{\varepsilon}{r}\partial_v(\langle\partial_u,\partial_t\rangle\langle\partial_v,\partial_t\rangle)\\
&= -2H\frac{\varepsilon}{r}e^{2\omega}\langle\partial_u^h,n_1^h\rangle - \frac{\varepsilon}{r}\partial_v(\langle\partial_u,\partial_t\rangle\langle\partial_v,\partial_t\rangle)\\
&= -2H\frac{\varepsilon}{r}e^{2\omega}\langle\partial_u^h,n_1^h\rangle + \partial_v\Im\psi^2.
\end{aligned}$$

We have used the formula $\Delta t = 2H\langle n_1,\partial_t\rangle$, where Δ is the Laplacian on Σ (see Section 6). Similarly, we prove that

$$\partial_v\Re\psi^2 = -\partial_u\Im\psi^2 + 2H\frac{\varepsilon}{r}e^{2\omega}\langle\partial_v^h,n_1^h\rangle.$$

Then, using that $\kappa = \epsilon/r^2$ (as remarked previously), we conclude that the function $\psi := 2H\psi^1 - \varepsilon\frac{\epsilon}{r}\psi^2$ satisfies

$$\begin{aligned}
\partial_u\Re\psi &= \partial_v\Im\psi + 2\Re\psi^1 H_u - 2\Im\psi^1 H_v + 2e^{2\omega}HH_u\\
&= \partial_v\Im\psi + 2eH_u + 2fH_v,\\
\partial_v\Re\psi &= -\partial_u\Im\psi + 2\Re\psi_1 H_v + 2\Im\psi^1 H_u - 2e^{2\omega}HH_v\\
&= -\partial_u\Im\psi - 2gH_v - 2fH_u.
\end{aligned}$$

Now, using the complex parameter $z = u + iv$ and the complex derivation $\partial_{\bar z} = \frac{1}{2}(\partial_u + i\partial_v)$, we get

$$\begin{aligned}
\partial_{\bar z}\psi &= (\partial_u\Re\psi - \partial_v\Im\psi) + i(\partial_v\Re\psi + \partial_u\Im\psi)\\
&= 2eH_u + 2fH_v - 2ifH_u - 2igH_v.
\end{aligned}$$

That is, defining the quadratic differential $Q := 2H\Psi^1 - \varepsilon\frac{\epsilon}{r}\Psi^2$, we prove that Q is holomorphic on Σ if H is constant. Inversely, if Q is holomorphic then

$$eH_u + fH_v = 0 \quad\text{and}\quad fH_u + gH_v = 0.$$

This implies that $A\nabla H = 0$, where $A = \langle \mathrm{d}X,\mathrm{d}X\rangle^{-1}\langle \mathrm{d}n_1,\mathrm{d}X\rangle$ is the shape operator for X and ∇H is the gradient of H on Σ. If we suppose that H is not constant, then $\nabla H \neq 0$ on an (open) set Σ'. On Σ' we have that $K_{\text{ext}} =: \det A = 0$ and $e_1 =: \nabla H/|\nabla H|$ is a principal direction with principal curvature $\kappa_1 = 0$. We also have $H = \kappa_2$, where κ_2 is the principal curvature of Σ calculated on a direction e_2 perpendicular to e_1. Hence, the only planar (umbilical) points on Σ' are the points where H vanishes. And the integral curves of e_2, because they are orthogonal to ∇H, are level curves for $H = \kappa_2$. Thus, H is constant along each such line.

THEOREM 4. *The quadratic differential $Q = 2H\Psi^1 - \varepsilon\frac{\epsilon}{r}\Psi^2$ is holomorphic on Σ if H is constant.*

The foregoing considerations indicate that, if there exist examples of surfaces with holomorphic Q and nonconstant mean curvature, then these examples must (a) be noncompact, (b) have zero extrinsic Gaussian curvature, and (c) be foliated by curvature lines along which H is constant. P. Mira and I. Fernández have informed the authors that they have constructed such examples.

For $\varepsilon = 1$, the quadratic form Q coincides with that obtained by Abresch and Rosenberg in [1]. It is clear that Q is the $(2,0)$ part of the complexification of the traceless part of the second fundamental form q corresponding to the normal direction $2Hn_1 - \varepsilon\frac{\varepsilon}{r}n_2$ on the normal bundle of $\Sigma \looparrowright \mathbb{E}^4$.

Using Theorem 4, we present the following generalization of the theorem of Abresch and Rosenberg quoted in Section 1.

THEOREM 5. *Let $X\colon \Sigma \to \mathbb{M}^2(\kappa) \times \mathbb{R}$ be a complete CMC immersion of a surface Σ in $\mathbb{M}^2(\kappa) \times \mathbb{R}$. If $\varepsilon = 1$ and Σ is homeomorphic to a sphere, then $X(\Sigma)$ is a rotationally invariant spherical surface. If Σ is homeomorphic to a disc and $Q \equiv 0$ on Σ, then $X(\Sigma)$ is a rotationally invariant disc. For $\varepsilon = -1$ and $\kappa \leq 0$, if $X(\Sigma)$ is simply connected and spacelike and if $Q \equiv 0$ on Σ, then the same conclusion holds.*

Proof. By hypothesis, we have $Q \equiv 0$ (if Σ is homeomorphic with a sphere, then this follows because Q is holomorphic). Thus, $2H\psi^1 \equiv \varepsilon\frac{\epsilon}{r}\psi^2$. Given an arbitrary local orthonormal frame field $\{e_1, e_2\}$, we may write this as

$$2Hh^1_{12} = \kappa\langle e_1, \partial_t\rangle\langle e_2, \partial_t\rangle, \tag{23}$$

$$2H(h^1_{11} - h^1_{22}) = \kappa\langle e_1, \partial_t\rangle^2 - \kappa\langle e_2, \partial_t\rangle^2. \tag{24}$$

If $H = 0$, then it follows from these equations that the vector field ∂_t is always normal to Σ. Therefore, the surface is a plane $\mathbb{P}_t = \mathbb{M}^2(\kappa) \times \{t\}$ for some $t \in \mathbb{R}$.

We thus need consider only CMC surfaces with $H \neq 0$. If (p,t) is an umbilical point of Σ then, for an arbitrary frame, we have $h^1_{12} = 0$ at this point. So either $\langle e_1, \partial_t\rangle = 0$ or $\langle e_2, \partial_t\rangle = 0$ at (p,t). Since $h^1_{11} = h^1_{22} = H$ at (p,t), equation (24) implies that both angles $\langle e_i, \partial_t\rangle$ are null. We conclude that if $Q = 0$ then umbilical points are the points where Σ has horizontal tangent plane, and vice versa.

If (p,t) is not an umbilical point in Σ, then we may choose $\{e_1, e_2\}$ as the principal frame on a neighborhood Σ' of that point. Thus, $h^1_{12} = 0$ and so $\langle e_1, \partial_t\rangle = 0$ or $\langle e_2, \partial_t\rangle = 0$ on Σ'. We fix $\langle e_1, \partial_t\rangle = 0$. If we denote by τ the tangential part

$\partial_t - \varepsilon\langle\partial_t, n_1\rangle n_1$ of the field ∂_t, then $\tau = \langle e_2, \partial_t\rangle e_2$. It follows from (24) that the principal curvatures of Σ are

$$h_{11}^1 = H - \frac{\kappa}{4H}|\tau|^2 \quad \text{and} \quad h_{22}^1 = H + \frac{\kappa}{4H}|\tau|^2.$$

The lines of curvature on Σ' with direction e_1 are locally contained in the planes $\mathbb{P}_t$. Conversely, the connected components of $\Sigma' \cap \mathbb{P}_t$ are lines of curvature with tangent direction given by e_1. Thus, if we parameterize such a line by its arc length s, then

$$\frac{\mathrm{d}}{\mathrm{d}s}\langle n_1, \partial_t\rangle = \langle D_{e_1} n_1, \partial_t\rangle = h_{11}^1\langle e_1, \partial_t\rangle = 0. \tag{25}$$

We conclude that, for a fixed t, Σ' and $\mathbb{P}_t$ make a constant angle $\theta(t)$ along each connected component of their intersection. Therefore, if a connected component of the intersection between $\mathbb{P}_t$ and Σ has a nonumbilical point, then the angle is constant and nonzero along this component unless there also exists an umbilical point on this same component. However, at this point the angle is necessarily zero. So by continuity of the angle function, either all points on a connected component $\Sigma \cap \mathbb{P}_t$ are umbilical and the angle is zero, or all points are nonumbilical and the angle is nonzero. But suppose that all points on a connected component σ are umbilical points for h^1. Then, as we noted previously, Σ is tangent to $\mathbb{P}_t$ along σ. So, along σ, we have $\langle e_1, \partial_t\rangle = \langle e_2, \partial_t\rangle = 0$ and thus, by equations (23) and (24), $h_{ii}^1 = 0$ and $H = 0$. From this contradiction, we conclude that the umbilical points may not be on *any* curve on $\Sigma \cap \mathbb{P}_t$. The only possibility is that there exist isolated umbilical points, as may occur on the top and bottom levels $t = a$ and $t = b$ of $X(\Sigma)$.

Hence there exists an orthonormal principal frame field $\{e_1, e_2\}$ on a dense subset of Σ. On this dense subset we have $\tau \neq 0$ and we may choose a positive sign for $\sin\theta(t)$ or $\sinh\theta(t)$, where $\theta(t)$ is the angle between n_1 and ∂_t along a given component of $\Sigma \cap \mathbb{P}_t$. We denote both of these functions by the same symbol $\mathrm{sn}(t)$. Now, we calculate the geodesic curvature of the horizontal curvature lines on $\mathbb{P}_t$. We have

$$e_2 = \frac{\tau}{|\tau|} = \frac{1}{\mathrm{sn}(t)}\tau = \frac{1}{\mathrm{sn}(t)}(\partial_t - \varepsilon\langle\partial_t, n_1\rangle n_1) = \frac{1}{\mathrm{sn}(t)}(\partial_t - \dot{\mathrm{sn}}(t) n_1).$$

Since $\langle n_1, \partial_t\rangle$ is constant along this curve and thus $\mathrm{sn}(t)$ is constant, we conclude that

$$D_{e_1} e_2 = \frac{1}{\mathrm{sn}(t)}(D_{e_1}\partial_t - \dot{\mathrm{sn}}(t) D_{e_1} n_1) = \frac{\dot{\mathrm{sn}}(t)}{\mathrm{sn}(t)} h_{11}^1 e_1,$$

where $\dot{\mathrm{sn}}(t) = \cos\theta(t)$ for $\varepsilon = 1$ and $\dot{\mathrm{sn}}(t) = \cosh\theta(t)$ for $\varepsilon = -1$. So the geodesic curvature $\langle D_{e_1} e_1, e_2\rangle$ of the horizontal lines of curvature relative to Σ is given by $-(\dot{\mathrm{sn}}(t)/\mathrm{sn}(t)) h_{11}^1$. This means that the horizontal lines of curvature have constant geodesic curvature on Σ. Now, by defining $\nu = J e_1 = \varepsilon\,\mathrm{sn}(t) n_1 - \dot{\mathrm{sn}}(t) e_2$, we calculate

$$\langle D_{e_1}\nu, e_1\rangle = -\varepsilon\,\mathrm{sn}(t) h_{11}^1 - \dot{\mathrm{sn}}(t)\frac{\dot{\mathrm{sn}}(t)}{\mathrm{sn}(t)} h_{11}^1 = -\frac{1}{\mathrm{sn}(t)} h_{11}^1.$$

It follows that the geodesic curvature of the horizontal lines of curvature on $\Sigma \cap \mathbb{P}_t$ relative to the plane $\mathbb{P}_t$ is also constant and equal to $h^1_{11}/\mathrm{sn}(t)$. We conclude that, for each t, $\Sigma \cap \mathbb{P}_t$ consists of constant geodesic curvature lines of $\mathbb{P}_t$.

We also obtain $\langle D_{e_2}e_2, e_1\rangle = 0$, and so the curvature lines of Σ with direction e_2 are geodesics on Σ. We then prove that these lines are contained on vertical planes. Given a fixed point (p,t) in $\Sigma \cap \mathbb{P}_t$, and let $\alpha(s)$ be the line of curvature with $\alpha' = e_2$ passing by (p,t) at $s = 0$. We want to show that α is contained on the vertical geodesic plane Π determined by $e_2(p,t)$ and ∂_t; this is the plane spanned by e_2 and n_1 at (p,t). For each s, consider the vertical geodesic plane Π_s on $\mathbb{M}^2(\kappa) \times \mathbb{R}$ for which $e_2 = \alpha'(s)$ and $D_{e_2}e_2 = D_{\alpha'}\alpha'$ are tangent at $\alpha(s)$. This plane is of the form $\sigma_s \times \mathbb{R}$, where σ_s is some geodesic on $\mathbb{M}^2(\kappa)$ that in turn is the intersection of $\mathbb{M}^2(\kappa)$ and some plane π_s on $\mathbb{E}^3$ with unit normal $a(s)$. The intersection of the hyperplane $\pi_s \times \mathbb{R}$ of $\mathbb{E}^4$ with $\mathbb{M}^2(\kappa) \times \mathbb{R}$ is then the plane Π_s. Now $p(s) \wedge \alpha'(s) \wedge D_{\alpha'}\alpha'$ is a normal direction to that hyperplane on $\mathbb{E}^4$, where $p(s) = \alpha(s)^h$. However, since α is both a line of curvature and geodesic, it follows that

$$D_{\alpha'}\alpha' = D_{e_2}e_2 = (D_{e_2}e_2)^T + (D_{e_2}e_2)^N = (D_{e_2}e_2)^N = h^1_{22}n_1.$$

Thus we conclude that the unit normal to the hyperplane Π_s is

$$a(s) = p(s) \wedge e_2(s) \wedge n_1(s).$$

Differentiating yields $a' = 0$, so $a(s)$ is constant. Thus implies that $\Pi_s = \Pi$ for all s. Therefore, $\alpha(s)$ is a plane curve contained in Π. Notice that Π has normal $e_1(p,t)$ since $e_1(p,t) = a(0)$. We then conclude that the integral curves of e_2 are planar geodesics on Σ.

So, for a fixed t, let $\sigma(s)$ be a component of $\Sigma \cap \mathbb{P}_t$. Then σ is a constant geodesic curvature curve on $\mathbb{P}_t$. Moreover, the vertical plane passing through $\sigma(s)$ with normal $e_1(\sigma(s))$ is a symmetry plane of Σ because it contains a geodesic of Σ—namely, the curvature line in direction e_2 passing through $\sigma(s)$. Thus, the surface is invariant with respect to the isometries fixing σ. Since the surface is homeomorphic to a disc or a sphere (see Remark 2), we conclude that these isometries are elliptic (their orbits are closed circles). This means that $X(\Sigma)$ is rotationally invariant in the sense of Section 1, which concludes the proof of Theorem 5. □

Remark 1. We can also prove Theorem 5 by reasoning as follows. Denote by Π_s the plane passing through $\sigma(s)$ with normal e_1. This plane contains the curvature line with initial data $\sigma(s)$ for position and $e_2(\sigma(s))$ for velocity, and its plane curvature is given by the derivative of its angle with respect to the (fixed) direction ∂_t—that is, $\theta(t)$. These data, by the fundamental theorem on planar curves, completely determine the curve. Changing the point on σ, the initial data differ by a rigid motion (an isometry on $\mathbb{P}_t$) and the curvature function remains the same at points of equal height. So, the two curves differ only by the same rigid motion. This means that the surface is invariant by the rigid motions fixing σ. Thus, the proof is finished by proving that the only possible isometries are the elliptic ones.

Remark 2. For $\kappa \leq 0$ and $\varepsilon = -1$, since $X(\Sigma)$ is spacelike, it is acausal. Thus, the coordinate t is bounded on Σ. Moreover, the projection $(p,t) \in \Sigma \mapsto p \in \mathbb{M}^2(\kappa)$ increases Riemannian distances and so is a covering map; therefore, $X(\Sigma)$

is locally a graph over the horizontal planes. If we suppose Σ to be simply connected, then $X(\Sigma)$ is globally diffeomorphic with $\mathbb{P}_t$; in fact, $X(\Sigma)$ is a disc-type graph.

Let $X\colon \Sigma \to \mathbb{M}^2(\kappa)\times\mathbb{R}$ be an immersion of a surface with boundary. We suppose that $X|_{\partial\Sigma}$ is a diffeomorphism onto its image $\Gamma = X(\partial\Sigma)$. We further suppose that $X(\partial\Sigma)$ is contained on some plane $\mathbb{P}_t$. Thus, Γ is an embedded curve on $\mathbb{P}_t$ that bounds a domain Ω. In what follows we always make this hypothesis while treating immersions of surfaces with boundary. Now fix $\varepsilon = -1$ and suppose that $X(\Sigma)$ is spacelike. We may prove that, under these assumptions, Σ is simply connected (disc type) and $X(\Sigma)$ is a graph over Ω. This conclusion also holds if Γ is supposed to be a graph over some embedded curve on $\mathbb{P}_t$.

Therefore, if we suppose either $\varepsilon = 1$ and Σ a disc or $\varepsilon = -1$ (with the additional hypothesis in both cases that $Q = 0$), then we are able to prove that if $X(\Sigma)$ is an immersed CMC surface with boundary then $X(\Sigma)$ is contained on a rotationally invariant CMC disc. In fact, the reasoning used to prove Theorem 5 works well in these cases to show that $X(\Sigma)$ is foliated by geodesic circles and that the angle with a plane $\mathbb{P}_t$ is constant along $\Sigma\cap\mathbb{P}_t$. This suffices to show that $X(\Sigma)$ is rotationally invariant.

5. Free Boundary Surfaces in Product Spaces

A classical result of J. Nitsche (see e.g. [13; 16; 17]) characterizes discs and spherical caps as equilibria solutions for the free boundary problem in space forms. We will be concerned now about how to reformulate this problem in the product spaces $\mathbb{M}^2(\kappa)\times\mathbb{R}$.

Let Σ be an orientable compact surface with nonempty boundary and let $X\colon\Sigma\to\mathbb{M}^2(\kappa)\times\mathbb{R}$ be an isometric immersion. By a *volume-preserving variation of* X we mean a family $X_s\colon\Sigma\to\mathbb{M}^2(\kappa)\times\mathbb{R}$ of isometric immersions such that $X_0 = X$ and $\int\langle\partial_s X_s, n_s\rangle\,\mathrm{d}A_s = 0$, where $\mathrm{d}A_s$ and n_s represent respectively the element of area and an unit normal vector field to X_s. In the sequel we set $\xi = \partial_s X_s$ and $f = \langle\xi_s, n_s\rangle$ at $s = 0$. We say that X_s is an *admissible variation* if it is volume-preserving and if, at each time s, the boundary $X_s(\partial\Sigma)$ of $X_s(\Sigma)$ lies on a horizontal plane $\mathbb{P}_a$. We denote by Ω_s the compact domain in $\mathbb{P}_a$ whose boundary is $X_s(\partial\Sigma)$ (in the spherical case $\kappa > 0$, we choose one of the two domains bounded by $X_s(\partial\Sigma)$). A *stationary surface* is by definition a critical point for the functional

$$E(s) = \int_\Sigma \mathrm{d}A_s + \alpha\int_{\Omega_s}\mathrm{d}\Omega$$

for some constant α, where $\mathrm{d}\Omega$ is the volume element for Ω_s induced from $\mathbb{P}_a$. The first variation formula for this functional (see [5; 17] for the corresponding formulas in space forms) is

$$E'(0) = -2\int_\Sigma Hf + \int_{\partial\Sigma}\langle\xi, \eta + \alpha\bar\eta\rangle\,\mathrm{d}\sigma,$$

where $\mathrm{d}\sigma$ is the line element for $\partial\Sigma$ and $\eta, \bar\eta$ are the unit co-normal vector fields to $\partial\Sigma$ relative to Σ and to $\mathbb{P}_a$. If we prescribe $\alpha = -\cos\theta$ in the Riemannian case and $\alpha = -\cosh\theta$ in the Lorentzian case, then we conclude that a stationary surface Σ has constant mean curvature and makes constant angle θ along $\partial\Sigma$ with the horizontal plane.

In what follows, *spherical cap* means that the surface is a part of a CMC revolution sphere bounded by some circle contained in a horizontal plane and centered at the rotation axis. Similarly, the term *hyperbolic cap* means a part of a CMC rotationally invariant disc bounded by a horizontal circle centered at the rotation axis. Granted this, we state the following theorem.

THEOREM 6. *Let Σ be a surface with boundary and let $X\colon \Sigma \to \mathbb{M}^2(\kappa) \times \mathbb{R}$ be a* stationary *immersion for free boundary admissible variations whose boundary lies in some plane $\mathbb{P}_a$. If $\varepsilon = 1$ and Σ is disc-type, then $X(\Sigma)$ is a spherical cap. If $\varepsilon = -1$, then $X(\Sigma)$ is a hyperbolic cap.*

Proof. The proof of Theorem 6 follows closely the guidelines of the proof of Nitsche's theorem in $\mathbb{R}^3$ (see [13; 17]). Let Σ denote the disc $|z| < 1$ in $\mathbb{R}^2$, where $z = u + iv$. If we put $\partial_z = \frac{1}{2}(\partial_u - i\partial_v)$, then the $\mathbb{C}$-bilinear complexification of q satisfies

$$q_{\mathbb{C}}(\partial_z, \partial_z) = q(\partial_u, \partial_u) - q(\partial_v, \partial_v) - 2iq(\partial_u, \partial_v) = 2Q(\partial_z, \partial_z).$$

Now, since $X(\partial\Sigma)$ is contained in $\mathbb{P}_a$, it follows that $q(\tau, \eta) = 0$ on $\partial\Sigma$. Here $\tau = e^{-\omega}(-v\partial_u + u\partial_v)$ is the unit tangent vector to $\partial\Sigma$ and $\eta = e^{-\omega}(u\partial_u + v\partial_v)$ is the unit outward co-normal to $\partial\Sigma$. In fact, $h^2(\tau, \eta) = 0$ because τ is a horizontal vector and $h^1(\tau, \eta) = 0$ because $\partial\Sigma$ is a line of curvature for Σ (by Joachimstahl's theorem).

On the other hand, on $\partial\Sigma$ we have that

$$0 = q(\tau, \eta) = (u^2 - v^2)q(\partial_u, \partial_v) - uvq(\partial_u, \partial_u) + uvq(\partial_v, \partial_v) = \Im(z^2 Q(\partial_z, \partial_z)).$$

From this we conclude that $\Im(z^2 Q) \equiv 0$ on $\partial\Sigma$. Since $z^2 Q$ is holomorphic on Σ, we know that $\Im z^2 Q$ must be harmonic. Hence $\Im z^2 Q = 0$ on Σ and so $z^2 Q \equiv 0$ on Σ. Therefore, $Q \equiv 0$ on Σ. This implies that $X(\Sigma)$ is part of a CMC revolution sphere or a CMC rotationally invariant disc, which finishes the proof of Theorem 6. □

We also obtain a result about stable CMC discs in $\mathbb{M}^2(\kappa) \times \mathbb{R}$ by following ideas presented in [3]. Here, stability for a CMC surface Σ means that the quadratic form

$$J[f] = \varepsilon \int_{\Sigma} (\Delta f + \varepsilon(|A|^2 + \mathrm{Ric}(n_1, n_1)) f) f \, \mathrm{d}A$$

is nonnegative with respect to all the variational fields f that generate volume-preserving variations (see [6] and [7] for the case $\kappa = 0$). In the preceding formula, Ric means the Ricci curvature tensor of $\mathbb{M}^2(\kappa) \times \mathbb{R}$.

THEOREM 7. *Let Σ be an immersed surface with boundary and constant mean curvature H in $\mathbb{M}^2(\kappa) \times \mathbb{R}$. Suppose that $\partial\Sigma$ is a geodesic circle in some plane $\mathbb{P}_a$ and that the immersion is stable. We further suppose that Σ is disc-type for*

$\varepsilon = 1$ and that the immersion is spacelike for $\varepsilon = -1$. Then Σ is a spherical or hyperbolic cap if $H \neq 0$. If $H = 0$ then Σ is a totally geodesic disc.

Proof. We consider the vector field $Y(t,p) = a \wedge \partial_t \wedge p$, where a is the vector in $\mathbb{E}^3$ that is perpendicular to the plane where $\partial\Sigma$ lies. This is a Killing field in $\mathbb{M}^2(\kappa) \times \mathbb{R}$. Then $f = \langle Y, n_1 \rangle$ trivially satisfies $J[f] = 0$. Let η be the exterior unit co-normal direction to Σ along the boundary $\partial\Sigma$.

The normal derivative of f along $\partial\Sigma$ is calculated as

$$\begin{aligned}\eta(f) = \eta\langle Y, n_1\rangle &= \langle a \wedge \partial_t \wedge D_\eta p, n_1\rangle + \langle a \wedge \partial_t \wedge p, D_\eta n_1\rangle \\ &= \langle a \wedge \partial_t \wedge \eta, n_1\rangle + \langle a \wedge \partial_t \wedge p, D_\eta n_1\rangle \\ &= -\langle a \wedge \partial_t \wedge n_1, \eta\rangle + \langle a \wedge \partial_t \wedge p, D_\eta n_1\rangle \\ &= \langle \tau, \eta\rangle + \langle \tau, D_\eta n_1\rangle = \langle \tau, D_\eta n_1\rangle = -h^1(\tau, \eta),\end{aligned}$$

where $\tau = a \wedge \partial_t \wedge p$ (the restriction of Y to the boundary of Σ) is the tangent positively oriented unit vector to $\partial\Sigma$. Since $\langle \tau, \partial_t\rangle = 0$ and $\langle \tau, \eta\rangle = 0$, it follows that

$$h^2(\tau, \eta) = -\frac{1}{r}\langle \tau^h, \eta^h\rangle = 0.$$

This yields

$$2H\eta(f) = -2Hh^1(\tau, \eta) = -q(\tau, \eta).$$

However, if u, v denote the usual Cartesian coordinates on Σ then

$$q(\tau, \eta) = e^{-2\omega} q(u\partial_u + v\partial_v, -v\partial_u + u\partial_v) = -\Im(z^2 Q)$$

on $\partial\Sigma$. We conclude that $2H\eta(f) = \Im(z^2 Q)$. Proceeding as in [3], we verify that $\eta(f)$ vanishes at least three times. Applying Courant's theorem on nodal domains allows us to conclude that f vanishes on the whole disc. Hence $X(\Sigma)$ is foliated by the flux lines of Y—that is, by horizontal geodesic circles centered at the same vertical axis. Therefore, $X(\Sigma)$ is a spherical or hyperbolic cap as we claimed. This proves Theorem 7. □

6. Flux Formula and Geometric Estimates

Let Σ be an immersed surface in $\mathbb{M}^2(\kappa) \times \mathbb{R}$ with constant mean curvature H relative to a unit normal vector field n. Consider a Killing vector field Y on $\mathbb{M}^2(\kappa) \times \mathbb{R}$. Thus restricting Y to Σ, one obtains the *flux formula* for Killing vector fields:

$$\int_{\partial\Sigma} \langle Y, \eta\rangle \, \mathrm{d}\sigma + 2H\varepsilon \int_\Omega \langle Y, n_\Omega\rangle \, \mathrm{d}\Omega = 0, \tag{26}$$

where Ω is an oriented surface homologous to Σ on $\mathbb{M}^2(\kappa) \times \mathbb{R}$, η is the outward unit co-normal to Σ along its boundary, and n_Ω is the unit normal to Ω such that the cycle $\Sigma \cup \Omega$ is coherently oriented.

The restriction of a Killing field to a surface is a Jacobi field. Then we have

$$(\Delta + \varepsilon|A|^2 + \varepsilon \operatorname{Ric}(n,n))\langle Y, n\rangle = 0.$$

Hence, using the expression

$$\mathrm{Ric}(n,n) = \kappa\varepsilon(1 - \langle n, \partial_t\rangle^2),$$

we conclude that

$$\Delta\langle Y, n\rangle = -\varepsilon(|A|^2 + \kappa\varepsilon(1 - \langle n, \partial_t\rangle^2))\langle Y, n\rangle.$$

We now consider the Killing field $Y = \partial_t$. It is clear that the gradient of t restricted to Σ is $\nabla t = \varepsilon\partial_t^T$ and that its Laplacian is given by

$$\Delta t = 2H\langle\partial_t, n\rangle.$$

We then fix $\varepsilon = -1$. Suppose the boundary of Σ is a geodesic circle in some horizontal plane $\mathbb{P}$ given by $t = 0$. We choose an upward orientation to Σ so that $\langle\partial_t, n\rangle \leq 0$ on Σ. In this case, we have $H \leq 0$. Thus, denoting $\nu = \langle\partial_t, n\rangle$, the function $\phi =: Ht - \nu$ satisfies

$$\Delta\phi = (2H^2 - |A|^2 + \kappa(1 - \nu^2))\nu. \tag{27}$$

One may verify that

$$|A|^2 = 2H^2 + 2|\psi^1|^2 \quad \text{and} \quad 4|\psi^2|^2 = \kappa^2(1 - \nu^2)^2;$$

since $\kappa \leq 0$ and $1 - \nu^2 \leq 0$, we have $2|\psi^2| = \kappa(1 - \nu^2)$. Substituting this into (27) and assuming that $|\psi^1|^2 - |\psi^2| \geq 0$ yields

$$\Delta\phi = -2(|\psi^1|^2 - |\psi^2|)\nu \geq 0.$$

Then, by Stokes's theorem,

$$-2\int_\Sigma (|\psi^1|^2 - |\psi^2|)\nu \,\mathrm{d}A = \int_{\partial\Sigma} \langle\nabla\phi, \eta\rangle \,\mathrm{d}\sigma,$$

where η is the outward unit co-normal to Σ along $\partial\Sigma$. However,

$$\langle\nabla\phi, \eta\rangle = H\langle\nabla t, \eta\rangle - \langle\nabla\nu, \eta\rangle = -H\langle\partial_t, \eta\rangle + \langle\partial_t, A\eta\rangle.$$

As a result, $\langle\nabla\phi, \eta\rangle = (\langle A\eta, \eta\rangle - H)\langle\eta, \partial_t\rangle$. But $\langle A\eta, \eta\rangle = 2H - \langle A\tau, \tau\rangle$, where τ is the unit tangent vector to $\partial\Sigma$. Let $\bar{\eta}$ be the outward unit normal to $\partial\Sigma$ with respect to $\mathbb{P}$. Since $n = \langle n, \bar{\eta}\rangle\bar{\eta} - \langle n, \partial_t\rangle\partial_t$ and since τ is orthogonal to both ∂_t and $\bar{\eta}$, it follows that

$$-\langle A\tau, \tau\rangle = \langle D_\tau n, \tau\rangle = \langle n, \bar{\eta}\rangle\langle D_\tau\bar{\eta}, \tau\rangle = -\kappa_g\langle n, \bar{\eta}\rangle = \kappa_g\langle\partial_t, \eta\rangle.$$

Thus we conclude that $\langle\nabla\phi, \eta\rangle = (H + \kappa_g\langle\eta, \partial_t\rangle)\langle\eta, \partial_t\rangle$. So, by the flux formula we have

$$\begin{aligned}\int_{\partial\Sigma} \langle\nabla\phi, \eta\rangle \,\mathrm{d}\sigma &= H\int_{\partial\Sigma} \langle\eta, \partial_t\rangle \,\mathrm{d}\sigma + \int_{\partial\Sigma} \kappa_g\langle\eta, \partial_t\rangle^2 \,\mathrm{d}\sigma \\ &= 2H^2|\Omega| + \int_{\partial\Sigma} \kappa_g\langle\eta, \partial_t\rangle^2 \,\mathrm{d}\sigma.\end{aligned}$$

Gathering the expressions then yields

$$-2\int_\Sigma (|\psi^1|^2 - |\psi^2|)\nu \,\mathrm{d}A = 2H^2|\Omega| + \int_{\partial\Sigma} \kappa_g\langle\eta, \partial_t\rangle^2 \,\mathrm{d}\sigma.$$

Now, again by the flux formula,

$$\left(\int_{\partial\Sigma}\langle\eta,\partial_t\rangle\,\mathrm{d}\sigma\right)^2 = 4H^2|\Omega|^2;$$

but by Cauchy–Schwarz on L^2 functions we have

$$\left(\int_{\partial\Sigma}\langle\eta,\partial_t\rangle\,\mathrm{d}\sigma\right)^2 \le |\partial\Sigma|\int_{\partial\Sigma}\langle\eta,\partial_t\rangle^2\,\mathrm{d}\sigma.$$

Hence

$$\frac{4H^2|\Omega|^2}{|\partial\Sigma|} \le \int_{\partial\Sigma}\langle\eta,\partial_t\rangle^2\,\mathrm{d}\sigma.$$

Therefore,

$$-2\int_{\Sigma}(|\psi^1|^2-|\psi^2|)\nu\,\mathrm{d}A \le 2H^2\frac{|\Omega|}{|\partial\Omega|}(|\partial\Omega|+2|\Omega|\kappa_g), \tag{28}$$

with equality if and only if $\langle\eta,\partial_t\rangle$ is constant along $\partial\Sigma$.

Now, the geodesic curvature of $\partial\Sigma$ calculated with respect to $\bar{\eta}$ is $\kappa_g = -\mathrm{ct}_\kappa(\rho)$. Thus

$$|\partial\Omega| + 2|\Omega|\kappa_g = \frac{2\pi}{\kappa}\,\mathrm{sn}_\kappa(\rho)(\mathrm{cs}_\kappa(\rho)-1)^2 \le 0$$

since $\kappa \le 0$. So equality holds in (28). Then the angle between Σ and the horizontal plane is constant along $\partial\Sigma$. As a result, Σ is a stationary surface for the energy defined in Section 5. Thus, by Theorem 6, the surface is a hyperbolic cap.

THEOREM 8. *Fix $\varepsilon = -1$ and $\kappa \le 0$. Let Σ be a immersed CMC surface whose boundary is a geodesic circle on some horizontal plane $\mathbb{P}_t$. If we suppose that $|\psi^1|^2 - |\psi^2| \ge 0$, then $Q = 0$ and the surface is part of a hyperbolic cap or a planar disc.*

This theorem is a partial answer to a Lorentzian formulation of the well-known *spherical cap conjecture,* which was affirmed in [4] for the case $\kappa = 0$.

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