§ 3- sphere

1. 穩定的封閉 CMC 必為球面 J. L. Barbosa ＆ M. do Carmo 1984
2. 宇宙可能 99%是 S^3
3. Poincare 猜想 任何一個單連通 閉(closed) 3 維流形一定跟 S^3 拓撲等價

$$S^3, x^\mu = (\psi, \theta, \varphi)$$
$$ds^2 = d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2)$$

求

(a) $\Gamma^i_{jk} = \frac{1}{2} g^{il}\left\{\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right\}$

(b) 求 Riemannian tensor，Ricci tensor，Ricci scalar

$$R^\rho_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

$$R^l_{ijk} = \Omega^l_i(E_j, E_k)$$

(c) Show that $R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu} g\sigma\nu - g_{\rho\nu} g_{\sigma\mu})$ is obeyed by this metric，confirming that the 3-sphere is a maximally symmetric space。

$$E_1 = \frac{\partial}{\partial\psi}, E_2 = \frac{1}{\sin\psi}\frac{\partial}{\partial\theta}, E_3 = \frac{1}{\sin\psi\sin\theta}\frac{\partial}{\partial\phi}$$
$$\omega^1 = d\psi \text{ ， } \omega^2 = \sin\psi d\theta \text{ ， } \omega^3 = \sin\psi\sin\theta d\phi$$

Then $g = \sum_i \omega^i \otimes \omega^i$

Cartan formula：

$$d\omega^i = \sum_j \omega^j \wedge \omega^i_j \text{ ， } \omega^j_i + \omega^i_j = 0 \text{ ， } \Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j$$

$$d\omega^1 = 0, d\omega^2 = \cos\psi d\psi \wedge d\theta, d\omega^3 = \cos\psi\sin\theta d\psi \wedge d\phi + \sin\psi\cos\theta d\theta \wedge d\phi$$

$$d\omega^1 = \omega^2 \wedge \omega^1_2 + \omega^3 \wedge \omega^1_3 = 0 \Rightarrow d\theta \wedge \omega^2_1 + \sin\theta d\phi \wedge \omega^3_1 = 0$$

$$d\omega^2 = \omega^1 \wedge \omega^2_1 + \omega^3 \wedge \omega^2_3 = d\psi \wedge \omega^2_1 + \sin\psi\sin\theta d\phi \wedge \omega^2_3 = \cos\psi d\psi \wedge d\theta$$

$$\therefore \omega^2_1 = \cos\psi d\theta$$

$$d\omega^3 = \omega^1 \wedge \omega^3_1 + \omega^2 \wedge \omega^3_2 = \cos\psi\sin\theta d\psi \wedge d\phi + \sin\psi\cos\theta d\theta \wedge d\phi$$

$$= d\psi \wedge \omega_1^3 + \sin\psi d\theta \wedge \omega_2^3$$

$\therefore \omega_1^3 = \cos\psi \sin\theta d\phi$ ， $\omega_2^3 = \cos\theta d\phi$

$$\Omega_j^i = d\omega_j^i + \omega_k^{\ i} \wedge \omega_j^k$$

$$\Omega_1^2 = d\omega_1^2 - \omega_1^3 \wedge \omega_3^2 = -\omega^1 \wedge \omega^2$$

$$\Omega_1^3 = d\omega_1^3 - \omega_1^2 \wedge \omega_2^3 = -\omega^1 \wedge \omega^3$$

$$\Omega_2^3 = d\omega_2^3 - \omega_2^1 \wedge \omega_1^3 = -\omega^2 \wedge \omega^3$$

$R_{ijk}^l = \Omega_i^l(E_j, E_k)$ ， $R_{iji}^j = \Omega_i^j(E_j, E_i)$

$R_{212}^1 = \sin^2\psi, R_{221}^1 = -\sin^2\psi$ ， $R_{313}^1 = \sin^2\psi \sin^2\theta, R_{331}^1 = -\sin^2\psi \sin^2\theta$

$R_{121}^2 = \Omega_1^2(E_1, E_2) = 1$ ， $R_{112}^2 = -1$ ， $R_{323}^2 = \sin^2\psi \sin^2\theta, R_{332}^2 = -\sin^2\psi \sin^2\theta$

$R_{131}^3 = \Omega_1^3(E_1, E_3) = 1$ ， $R_{113}^3 = -1$ ， $R_{232}^3 = \Omega_2^3(E_2, E_3) = \sin^2\psi$ ， $R_{223}^3 = -\sin^2\psi$

$R_{\theta\psi\theta}^{\psi} = \dfrac{\partial \Gamma_{\theta\theta}^{\psi}}{\partial \psi} - \dfrac{\partial \Gamma_{\theta\psi}^{\psi}}{\partial \theta} + \Gamma_{\lambda\psi}^{\psi}\Gamma_{\theta\theta}^{\lambda} - \Gamma_{\lambda\theta}^{\psi}\Gamma_{\theta\psi}^{\lambda}$ ，where $\lambda = \psi, \theta, \phi$

$= \sin^2\psi$ has been confirmed。

Then $\mathrm{R}_{212}^1 = \Omega_2^1(E_1, E_2) = (\omega^1 \wedge \omega^2)(E_1, E_2) = ?$

$R_{ij} = \sum_k R_{ijk}^k$ or $\sum_k R_{ikj}^k$

$R_{ijkl} = -R_{jikl}$ ， $R_{ijkl} = -R_{ijlk}$ ， $R_{ijkl} = R_{kjij}$

$R_{11} = R_{111}^1 + R_{112}^2 + R_{113}^3 = 2$

$\mathrm{R}_{22} = R_{33} = 2$

$R = g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} = 6$

$R_{ij} = (n-1) g_{ij}$ where n=3， $g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2\psi & 0 \\ 0 & 0 & \sin^2\psi \sin^2\theta \end{pmatrix}$

The variational principle provides a convenient way to actually calculate the Christoffel symbols for a given metric。

$$I = \frac{1}{2}\int f d\tau = \frac{1}{2}\int g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} d\tau$$
$$ds^2 = d\psi^2 + \sin^2\psi (d\theta^2 + \sin^2\theta d\phi^2)$$
$$I = \frac{1}{2}\int [(\frac{d\psi}{d\tau})^2 + \sin^2\psi (\frac{d\theta}{d\tau})^2 + \sin^2\psi \sin^2\theta (\frac{d\phi}{d\tau})^2] d\tau$$

$S(q) = \int L(t, q(t), \dot{q}(t)) dt$ ， If it independent of t，then the E-L equation are

$\frac{\partial L}{\partial q^i} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{q}^i} = 0$ ，where $L = (\dot{\psi})^2 + \sin^2\psi (\dot{\theta})^2 + \sin\psi^2 \sin^2\theta (\dot{\phi})^2$

For ψ ，the E-L equation is $\frac{\partial L}{\partial \psi} - \frac{d}{d\tau}(\frac{\partial L}{\partial \dot{\psi}}) = 0$

$\frac{\partial L}{\partial \psi} = 2\sin\psi\cos\psi(\dot{\theta})^2 + 2\sin\psi\cos\psi\sin^2\theta(\dot{\phi})^2$ ，and $\frac{d}{d\tau}(\frac{\partial L}{\partial \dot{\psi}}) = \frac{d}{d\tau}(2\dot{\psi}) = 2\ddot{\psi}$

$2\sin\psi\cos\psi(\dot{\theta})^2 + 2\sin\psi\cos\psi\sin^2\theta(\dot{\phi})^2 - 2\ddot{\psi} = 0$ ，

$\ddot{\psi} - \sin\psi\cos\psi(\dot{\theta})^2 - \sin\psi\cos\psi\sin^2\theta(\dot{\phi})^2 = 0$

The geodesic equation are $\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0$ ，here $x^k = \psi$

The geodesic equation is $\ddot{\psi} + \Gamma^\psi_{\theta\theta}(\dot{\theta})^2 + \Gamma^\psi_{\phi\phi}(\dot{\psi})^2 = 0$

Thus we have $\Gamma^\psi_{\theta\theta} = -\sin\psi\cos\psi, \Gamma^\psi_{\phi\phi} = -\sin\psi\cos\psi\sin^2\theta$

For θ ，the Euler equation is $\frac{\partial L}{\partial \theta}-\frac{d}{d\tau}(\frac{\partial L}{\partial \dot{\theta}})=0$

$$\frac{\partial L}{\partial \theta}=2\sin^2\psi\sin\theta\cos\theta(\dot{\phi})^2$$

$$\frac{d}{d\tau}(\frac{\partial L}{\partial \dot{\theta}})=\frac{d}{d\tau}(\sin^2\psi(2\dot{\theta}))=2\sin\psi\cos\psi\,\dot{\psi}(2\dot{\theta})+2\sin^2\psi\,\ddot{\theta}$$

$$\ddot{\theta}+2\cot\psi\,\dot{\theta}\dot{\psi}-\sin\theta\cos\theta(\dot{\phi})^2=0$$

Compare with $\ddot{\theta}+\Gamma^{\theta}_{ij}\dot{x}^i\dot{x}^j=0$ ，

We have $\Gamma^{\theta}_{\psi\theta}=\Gamma^{\theta}_{\theta\psi}=\cot\psi,\Gamma^{\theta}_{\phi\phi}=-\sin\theta\cos\theta$

Again for ϕ ，the Euler equation is $\frac{\partial L}{\partial \phi}-\frac{d}{d\tau}(\frac{\partial L}{\partial \dot{\phi}})=0$ ，we have

$$\ddot{\phi}+2\cot\psi\,\dot{\psi}\dot{\phi}+2\cot\theta\,\dot{\theta}\dot{\phi}=0$$

So $\Gamma^{\phi}_{\psi\phi}=\Gamma^{\phi}_{\phi\psi}=\cot\psi,\Gamma^{\phi}_{\theta\phi}=\Gamma^{\phi}_{\phi\theta}=\cot\theta$

We have $\Gamma^{\psi}_{\theta\theta}=-\sin\psi\cos\psi,\Gamma^{\psi}_{\phi\phi}=-\sin\psi\cos\psi\sin^2\theta$

$$\Gamma^{\theta}_{\psi\theta}=\Gamma^{\theta}_{\theta\psi}=\cot\psi,\ \Gamma^{\theta}_{\phi\phi}=-\sin$$

$$\Gamma^{\phi}_{\psi\phi}=\Gamma^{\phi}_{\phi\psi}=\cot\psi,\ \Gamma^{\phi}_{\theta\phi}=\Gamma^{\phi}_{\phi\theta}=\cot\theta$$

(b) The Rieman tensor components are

$$R^{\rho}_{\sigma\mu\nu}=\partial_{\mu}\Gamma^{\rho}_{\nu\sigma}-\partial_{\nu}\Gamma^{\rho}_{\mu\sigma}+\Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma}-\Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

$R^{\psi}_{\theta\psi\theta}=\sin^2\psi$ ， $R^{\psi}_{\theta\theta\psi}=-\sin^2\psi$ ， $R^{\psi}_{\phi\psi\phi}=\sin^2\psi\sin^2\theta$ ， $R^{\psi}_{\phi\phi\psi}=-\sin^2\psi\sin^2\theta$

$R^{\theta}_{\psi\psi\theta}=-1$ ， $R^{\theta}_{\psi\theta\psi}=1$ ， $R^{\theta}_{\phi\theta\phi}=\sin^2\psi\sin^2\theta$ ， $R^{\theta}_{\phi\phi\theta}=-\sin^2\psi\sin^2\theta$

$R^{\phi}_{\psi\psi\phi}=-1$ ， $R^{\phi}_{\psi\phi\psi}=1$ ， $R^{\phi}_{\theta\theta\phi}=-\sin^2\psi$ ， $R^{\phi}_{\theta\phi\theta}=\sin^2\psi$

例如

$$\mathrm{R}^{\psi}_{\phi\psi\phi} = \partial_{\psi}\Gamma^{\psi}_{\phi\phi} - \partial_{\phi}\Gamma^{\psi}_{\psi\phi} + \Gamma^{\psi}_{\psi\lambda}\Gamma^{\lambda}_{\phi\phi} - \Gamma^{\psi}_{\phi\lambda}\Gamma^{\lambda}_{\psi\phi}$$

$$= \partial_{\psi}(-\sin\psi\cos\psi\sin^2\theta) - \partial_{\varphi}0 + 0 - (-\sin\psi\cos\psi\sin^2\theta)(\cot\psi)$$

$$= -\cos^2\psi\sin^2\theta + \sin^2\psi\sin^2\theta + \sin^2\psi\cos^2\theta = \sin^2\psi$$

$$R^{\theta}_{\psi\theta\psi} = \partial_{\theta}\Gamma^{\theta}_{\psi\psi} - \partial_{\psi}\Gamma^{\theta}_{\psi\theta} + \Gamma^{\theta}_{\theta\lambda}\Gamma^{\lambda}_{\psi\psi} - \Gamma^{\theta}_{\psi\lambda}\Gamma^{\lambda}_{\psi\theta}$$

$$= -\partial_{\psi}(\cot\psi) - (\cot\psi)(\cot\psi) = \csc^2\psi - \cot^2\psi = 1$$

$$R^{\phi}_{\theta\theta\phi} = \partial_{\theta}\Gamma^{\phi}_{\theta\phi} - \partial_{\phi}\Gamma^{\phi}_{\theta\theta} + \Gamma^{\phi}_{\theta\lambda}\Gamma^{\lambda}_{\theta\phi} - \Gamma^{\phi}_{\phi\lambda}\Gamma^{\lambda}_{\theta\theta}$$

$$= -\csc^2\theta + \cot^2\theta - \cot\psi(-\sin\psi\cos\psi) = -1 + \cos^2\psi = -\sin^2\psi$$

$R_{\psi\psi} = R^{\lambda}_{\psi\lambda\psi} = 1 + 1 = 2$ ， $R_{\theta\theta} = R^{\lambda}_{\theta\lambda\theta} = \sin^2\psi + \sin^2\psi = 2\sin^2\psi$

$$R_{\phi\phi} = R^{\lambda}_{\phi\lambda\phi} = \sin^2\psi\sin^2\theta + \sin^2\psi\sin^2\theta = 2\sin^2\psi\sin^2\theta$$

The Ricci tensor is twice the metric $R_{\mu\nu} = 2g_{\mu\nu}$

The Ricci scalar $R = g^{\mu\mu}\mathrm{R}_{\mu\mu} = 6$

(c)Show that $\mathrm{R}_{\rho\sigma\mu\nu} = \frac{\mathrm{R}}{n(n-1)}(g_{\rho\mu}g\sigma\nu - g_{\rho\nu}g_{\sigma\mu})$ is obeyed by this metric，confirming that the 3-sphere is a maximally symmetric space。

[Foundation of Differential Geometry VII] by Kobayashi Nomizu

A Riemannian manifold is called Einstein if $S = \rho g$ ，where S is the Ricci tensor and ρ is a constant。

p.35 Let M be a hypersurface immersed in R^{n+1} ，at each point of M，the Ricci tensor S is given by $S(X,Y) = g(AX,Y)traceA - g(A^2X,Y)$ ， $X, Y \in T_x(M)$

For $n \geq 3$ ，if M is Einstein then $\rho \geq 0$

(1) $\rho = 0 \cdots$

(2) $\rho > 0$ M is locally a hypersphere

參考

1. [Spacetime and Geometry] Ch3 EX08 and EX16
2. [RG4102]--- $I \times S^2$ $g = A^2(r)\, dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$
   S^2 做為一個 Riemannian manifold $g = d\theta^2 + \sin^2\theta d\phi^2$ induced from R^3
3. Wormhole metric by Ellis
   $c^2 d\tau^2 = d\rho^2 + (\rho^2 + n^2)(d\theta^2 + \sin^2\theta\, d\varphi^2)$ where n is the drainhole parameter
4. $ds^2 = -c^2 dt^2 + dl^2 + (b_0^2 + l^2)(d\theta^2 + \sin^2\theta d\varphi^2)$ by MT wormhole
5. Everything Wormhole
6. Ricci curvature