§ Ricci soliton

Let $(M^n, g(t))$ be a solution of the Ricci flow，and suppose $\varphi_t : M^n \to M^n$ is a time-dependent family of diffeomorphism with $\varphi_0 = id$ and $\sigma(t)$ is a time-dependent scale factor with $\sigma(0)=1$。

If $g(t) = \sigma(t)\varphi_t^* g(0) \cdots (*)$ then the solution $(M^n, g(t))$ is called **a Ricci soliton**。

(*) 兩邊微分後取 t=0

$$\frac{\partial}{\partial t} g(t) = \frac{d\sigma(t)}{dt}\varphi_t^* g(0) + \sigma(t)\frac{\partial}{\partial t}\varphi_t^* g(0)$$

$-2Ric(g(0)) = \sigma'(0)g(0) + L_V g(0) \cdots (*)$，where $V = \frac{d\varphi_t}{dt}$

A Ricci soliton structure is (M, g, X, λ)

$Ric(g) + \frac{1}{2}L_X g = \frac{\lambda}{2} g \cdots (**)$ 與(*)比較 $\lambda = -\sigma'(0)$

Tracing (**)，we have $R + divX = \frac{n\lambda}{2}$，R is the saclar curvature。

$$divX = tr(\nabla X) = \sum_{i=1}^{n} \nabla_i X^i$$

If f is a function，$\nabla f = df$，in local coordinates，$\nabla_i f := (df)_i = \frac{\partial f}{\partial x^i}$ and

$\nabla^i f := (\nabla f)^i = g^{ij}\nabla_j f$。

(**) simplifies to $Ric(g) + \nabla^2 f = \frac{\lambda}{2} g$ since $L_{\nabla f} g = 2\nabla^2 f$，here ∇^2 denote the Hessian。These are so-called **gradient Ricci solitons**。

[Remark]

Lie derivative of a form ω：

$X \in \chi(M)$，$L_X\omega := \lim_{t\to 0}\frac{1}{t}(\varphi_t^*\omega - \omega) = \frac{d}{dt}(\varphi_t^*\omega)\big|_{t=0}$，Where φ_t is the local flow of X。

The Lie derivative of the metric tensor g：

$(L_V g)_{ij} = V^k g_{ij,k} + V^k{}_{,i} g_{kj} + V^k{}_{,j} g_{ik}$ Or $(L_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu X^\rho + g_{\rho\mu}\partial_\nu X^\rho$

Note that，K is a Killing vector field $\Leftrightarrow L_K g = 0$

(M, g, X, λ) is a Ricci soliton，then $(M, g, K+X, \lambda)$ is also a Ricci soliton。

Lemma

On a Riemannian manifold (M,g)，we have $(L_X g)_{ij} = \nabla_i X_j + \nabla_j X_i$

Where ∇ denote the Levi-Civita connection of the metric g，for any vector field X。

Let ω be the 1-form due to the vector field X，$\omega(Y) = \langle X, Y \rangle$ then
$L_X g(Y,Z) = \ldots = (\nabla_Y \omega)(Z) + (\nabla_Z \omega)(Y)$
Let $\sigma'(0)=2\lambda$ in the result of Lemma 1.7 to write (*) in coordinates as

$$-2R_{ij} = 2\lambda g_{ij} + \nabla_i V_j + \nabla_j V_i$$

As a special case we can consider the case that V is the gradient vector field of some scalar function on M^n，i.e. $V_i = \nabla_u f$。The equation then becomes

$$R_{ij} + \lambda g_{ij} + \nabla_i \nabla_j f = 0$$

Such solutions are known as gradient Ricci solitons。
A gradient Ricci soliton is called shrinking if $\lambda < 0$，static if $\lambda = 0$，and expanding if $\lambda > 0$

§ Special and explicitly defined Ricci solitons

1. The Gaussian solitons
2. Shrinking round spheres

   The metrics of constant positive curvature on the sphere S^n are anturally shrinking gradient Ricci solitons，when paired with any constant potential function。

   If g_{S^n} is the round metric of constant sectional curvature equal to one，the rescaled metric $g = 2(n-1)g_{S^n}$ will satisfy [$Ric(g) + \nabla^2 f = \frac{\lambda}{2} g$]with the canonical choice of constant $\lambda = 1$。

   We call $(S^n, g, \frac{n}{2})$ the shrinking round sphere。

   Reference [Curve shorting flow]

3. Einstein manifolds $Ric(g) = \frac{\lambda}{2} g$ of constant scalar curvature $\frac{n\lambda}{2}$

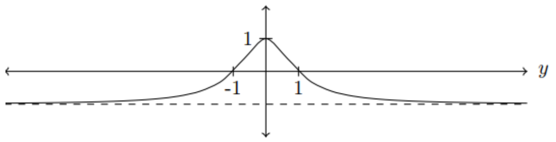
   $$Ric(g) + \frac{1}{2} L_X g = \frac{\lambda}{2} g$$

   If (M, g, X, λ) is Einstein soliton，then $L_X g = 0$
   The vector field X is Killing。
4. Product solitons

5. Quotient solitons
6. Nongradient(無梯度) solitons

The complete Riemannian metric $g = \frac{1}{1+y^2}(dx^2 + dy^2)$，together with the complete vector field $X = -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ generated by homotheties，comprises(包括) a complete nongradient expanding Ricci soliton struture $(R^2, g, X, -1)$ on R^2 。

The scalar curvature of g is given by R(x,y)=$\frac{1-y^2}{1+y^2}$



The scalar curvature as a function of y： $R(\cdot, y) = \frac{1-y^2}{1+y^2}$

Reference [hyperbolic plane] $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$

参考資料

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