§ Introduction to the Ricci flow

William Thurston 1946-2012

[On proof and Progress in Mathematics]

The connected sum decomposition

S^3, H^3, R^3

Hellmuth Kneser　　John Milor

Thurston Geometrization Conjecture

We have a Riemannian manifold M with the metric g_0，**the Ricci flow is a PDE** that evolves the metric tensor ：$\frac{\partial}{\partial t} g(t) = -2Ric(g(t))$，$g(0) = g_0$

A solution to this equation (or a Ricci flow) is a one-parameter family of metrics g(t)。

$(M, g(t_0))$ is called the initial condition (or initial metric)。

We hope that the metric will evolve towards one of the Thurston eight fundamental geometric structure。

In **harmonic coordinates** about p ($\Delta x^i = 0$ for all i)，we have

$Ric_{ij} = Ric(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = -\frac{1}{2}\Delta g_{ij} + Q_{ij}(g^{-1}, \partial g)$，where Q is a quadratic form in g^{-1} and ∂g，so in particular is a lower order term in the derivative of g。

Where Δ is the Laplace-Beltrami operator：$\Delta f = div(\nabla f)$。

So，in these coordinates，the Ricci flow equation is actually a **heat equation** for the Riemannian metric $\frac{\partial}{\partial t} g = \Delta g + 2Q(g^{-1}, \partial g)$

Theorem 2.1. *(Hamilton, 1982) Let $\mathcal{M}^3$ be a closed 3-manifold which admits a Riemannian metric with strictly positive Ricci curvature. Then $\mathcal{M}^3$ also admits a metric of constant positive curvature.*

Uniformization(單值化) Theorem：Any Riemannian metric on a closed 2-manifold is conformal to one of constant curvature。

任何單連通的黎曼曲面都共形等價於複平面、單位圓盤和黎曼球面三者之一。

§ Special Solution of the Ricci flow

Lemma 1.11

Let $X \in T_pM$ be a unit vector。Suppose that X is contained in some orthonormal basis for T_pM，$Ric(X, X)$ is then the sum of the sectional curvature of planes spanned by X and other elements of the basis。

Given an orthonormal basis for T_pM ，the scaar curvature at p is the sum of all sectional curature of planes spanned by pairs of basis elements。

For $S^n (n>1)$ of radius r，the metric is given $g = r^2 \bar{g}$ where $\bar{g}$ is the metric on the unit sphere。The sectional curvature are all $\frac{1}{r^2}$ 。

Thus for any unit vector v， $Ric(v,v) = \frac{n-1}{r^2}$ by Lemma 1.11。

Therefore $Ric = \frac{n-1}{r^2} g = (n-1)\bar{g}$

So the Ricci flow equation becomes an ODE

$$\frac{\partial g}{\partial t} = -2Ric(g) \Rightarrow \frac{\partial}{\partial t}(r^2 \bar{g}) = -2(n-1)\bar{g} \Rightarrow \frac{dr^2}{dt} = -2(n-1)$$

$r(t) = \sqrt{R_0{}^2 - 2(n-1)t}$ ，The manifold shrinks to a point as $t \to \frac{R_0{}^2}{2(n-1)}$ 。

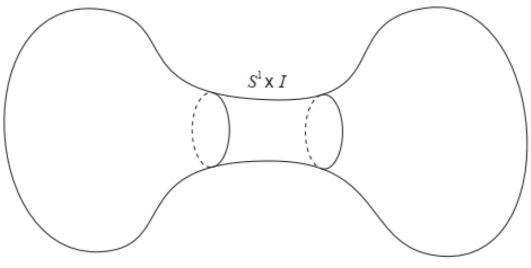
Similarly，for hyperbolic n-space $H^n (n>1)$ ，the Ricci flow reduces to the ODE

$\frac{d(r^2)}{dt} = 2(n-1)$ which has the solution $r(t) = \sqrt{R_0{}^2 + 2(n-1)t}$

So the solution expands out to infinity。

§ Pinching(捏)

How the connected sum decomposition arises out of Ricci flow？



Consider $S^1 \times I$ (I is an interval)between two parts of a 2-manifolds。

$S^2 \times I$

期待 Ch 8：The details of how the pinching off (by surgery 手術)actually happens will in Chapter 8。

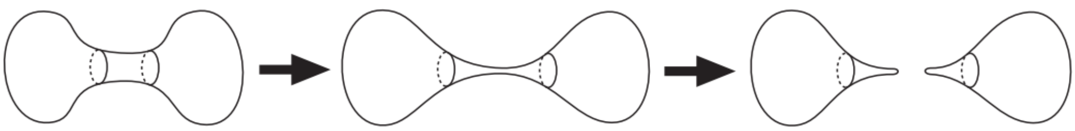


Figure 2.3: A neck "pinching off" in a 2-manifold. This diagram is intended to illustrate by lower-dimensional analogy what a neckpinch in a 3-manifold is like – the Ricci flow on 2-manifolds does not give rise to neckpinches.

The torus decomposition arises in a different way。
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1. An Illustrated introduction to the Ricci flow　Gabriel Khan　[部落格] 內有本書第三版
2.