Ricci flow and the Poincare conjecture   Gang Tian

§ Introduction to the Ricci flow

It is analogous to heat equation but it is non-lonear。

The Ricci flow was introduced by Richard Hamilton 1982。[ResearchGate]

[里奇流與 Poincare 猜想(1999 數學傳播 張樹城]

§ 01 Definition

We have a Riemannian manifold M with the metric g_0，**the Ricci flow is a PDE** that evolves the metric tensor：$\frac{\partial}{\partial t} g(t) = -2Ric(g(t))$，$g(0) = g_0$

A solution to this equation (or a Ricci flow) is a one-parameter family of metrics g(t)，$(\mathrm{M}, g(\mathrm{t}_0))$ is called the initial condition (or initial metric)。

We hope that the metric will evolve towards one of the Thurston eight fundamental geometric structure。

In **harmonic coordinates** about p，that is to say $\Delta x^i = 0$，we have

$R_{ij} = Ric(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = -\frac{1}{2}\Delta g_{ij} + Q_{ij}(g^{-1}, \partial g)$ where Q_{ij} is a quadratic form in g^{-1} and ∂g

So，the Ricci flow equation $\frac{\partial g}{\partial t} = -2Ric(g) = \Delta g + 2Q_{ij}(g^{-1}, \partial g)$ is a heat equation for the Riemannian metric。(heat equation $u_t = k\Delta u$)

Definition

The space-time for a Ricci flow is $M \times I$，where $t \in I$。

Given (p,t) and r>0，B(p,t,r) is the ball of radius r centered at (p,t) in the t time-slice。

The Laplacian：

1. $\Delta u = div(grad(u))$
2. Hessian matrix

$$\mathrm{Hess}(u) = \begin{bmatrix} \frac{\partial^2}{\partial x^2}u & \frac{\partial}{\partial x}\frac{\partial}{\partial y}u & \frac{\partial}{\partial x}\frac{\partial}{\partial z}u \\ \frac{\partial}{\partial y}\frac{\partial}{\partial x}u & \frac{\partial^2}{\partial y^2}u & \frac{\partial}{\partial y}\frac{\partial}{\partial z}u \\ \frac{\partial}{\partial z}\frac{\partial}{\partial x}u & \frac{\partial}{\partial z}\frac{\partial}{\partial y}u & \frac{\partial^2}{\partial z^2}u \end{bmatrix}$$

is called the Hessian matrix of u，then $\Delta u = trHess(u)$

the Ricci curvature is the trace of the Riemann curvature tensor。

[Ricci flow on S^2] 完全沒看懂。

$$\frac{\partial g}{\partial t}=(r-R(x,t))g(x,t), x\in M, t>0$$

Where R is the scalar curvature of g (= twice the Gaussian curvature K)，r is the average of R。The r in the equation is inserted to preserve the area of M。

§ 02 Some exact solutions to the Ricci flow

(1) Einstein manifolds

Let g_0 be an Einstein metric： $Ric(g_0)=\lambda g_0$ ，where λ is a constant。

Then for any constant c>0，setting $g=cg_0$

$$Ric(g)=Ric(cg_0)=Ric(g_0)=\lambda g_0=\frac{\lambda}{c}g$$

Consider $g(t)=u(t)g_0$ is the solution of the Ricci flow，then

$$\frac{\partial g}{\partial t}=u'(t)g_0=-2Ric(u(t)g_0)=-2Ric(g_0)=-2\lambda g_0$$

$\therefore u'(t)=-2\lambda, u(t)=1-2\lambda t$ ，thus $g(t)=(1-2\lambda t)g_0$ is a solution of the Ricci flow。

The case $\lambda>0, \lambda=0, \lambda<0$ correspond to shrinking，steady and expanding solutions。

Notice that in the ssrinking case the solution exists for $t\in[0,\frac{1}{2\lambda})$ and goes singular at $t=\frac{1}{2\lambda}$ 。

(2) The standard metric on each of S^n, R^n, H^n is Einstein。

(3) CP^n equipped with the Fubini-Study metric，which is induced from the standard metric of S^{2n+1} under the Hopf fibration with the fibers of great circles，is Einstein。

(4) Let h_0 be the round metric on S^2 with constant Gaussian curvature $\frac{1}{2}$ 。

Set $h(t)=(1-t)h_0$ ，then the flow $(S^2,h(t)),-\infty<t<1$ is a Ricci flow。

We also have the product of this flow with the trivial flow on the line $(S^2\times R, h(t)\times ds^2),-\infty<t<1$ 。This is called the standard shrinking round cylinder。

The standard shrinking round cylinder is a model for evolving $\varepsilon-necks$ 。

Definition

Let $(M,g(t))$ be a Ricci flow。An evolving $\varepsilon-neck$ centered at (x,t_0) and defined for rescaled time t_1 is an $\varepsilon-neck$

$\varphi:S^2\times(-\varepsilon^{-1},\varepsilon^{-1})\overset{\cong}{\rightarrow}N\subset(M,g(t))$ centered at (x,t_0) with the property that pull-back

via φ of the family of metric $R(x,t_0)g(t')|_N,-t_1<t'\le 0$

…

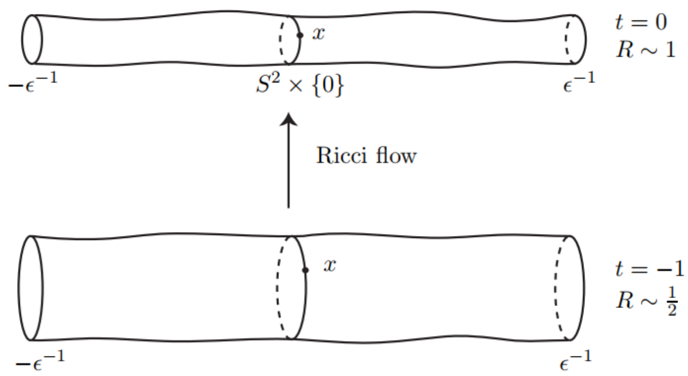


FIGURE 1. Strong ϵ-neck of scale 1.

A strong $\varepsilon-neck$ centered at (x,t_0) in a Ricci flow is an evolving $\varepsilon-neck$ centered at (x,t_0) and defined for rescaled time 1。As left。

§ [Ricci solitons]

A Ricci soliton is a Ricci flow，$0\le t<T\le\infty$，with the property that for each $t\in[0,T)$ there is a diffeomorphism $\varphi_t:M\rightarrow M$ and a constant $\sigma(t)$ such

$g(t)=\sigma(t)\varphi_t^*g(0)$

That is to say，in a Ricci soliton all the Riemannian manifold (M,g(t)) are isometric up to a scale factor that is allowed to vary with t。

The soliton is said to be shrinkng if $\sigma'(t)<0$ for all t。

Theorem Hamilton 1982

Let M^3 be a closed 3-manifold which admits a Riemannian metric with strictly positive Ricci curvature，then M^3 also admits a metric of constant positive curvature。

Uniformization(單值化) Theorem：

Any Riemannian metric on a closed 2-manifold is conformal to one of constant curvature。

任何單連通的黎曼曲面都共形等價於複平面、單位圓盤和黎曼球面三者之一。
Whether there is a natural evolution equation which conformally deforms any metric on a surface to a constant curvature metric。

§ 03 Special Solution of the Ricci flow

Lemma 1.11

Let $X \in T_pM$ be a unit vector。Suppose that X is contained in some orthonormal basis for T_pM，$Ric(X,X)$ is then the sum of the sectional curvature of planes spanned by X and other elements of the basis。

Given an orthonormal basis for T_pM，the scalar curvature at p is the sum of all sectional curature of planes spanned by pairs of basis elements。

For $S^n (n>1)$ of radius r(t)，the metric is given $g = r^2 \overline{g}$，where $\overline{g}$ is the metric on the unit sphere。The sectional curvature are all $\frac{1}{r^2}$。

單位 3 維球 的 metric，$\overline{g} = ds^2 = d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2)$

半徑 r 的 3 維球，$g = ds^2 = r^2 d\psi^2 + r^2\sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2)$

此處半徑是時間的函數，$g = r^2\overline{g}$

n 維球的里奇張量 Ric(g)=(n-1)g，因此 Ricci flow 方程變成常微分方程

$$\frac{\partial g}{\partial t} = -2Ric(g) \Rightarrow \frac{\partial}{\partial t}(r^2\overline{g}) = -2(n-1)\overline{g} \Rightarrow \frac{dr^2}{dt} = -2(n-1)$$

$$r^2 = R_0^{\ 2} - 2(n-1)t$$

$r(t) = \sqrt{R_0^{\ 2} - 2(n-1)t}$，時間$t \to \frac{R_0^{\ 2}}{2(n-1)}$，此球縮為一點(稱為奇點 singularity)。

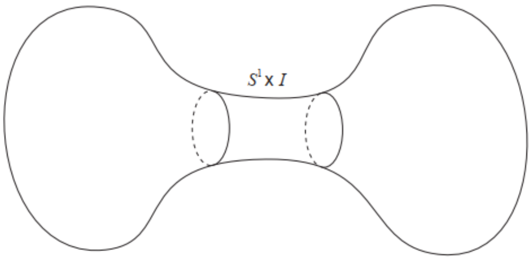
Similarly，for hyperbolic n-space $H^n(n>1)$，the Ricci flow reduces to the ODE

$\frac{d(r^2)}{dt} = 2(n-1)$ which has the solution $r(t) = \sqrt{R_0^{\ 2} + 2(n-1)t}$

So the solution expands out to infinity。

§ 04 Pinching(捏)

How the connected sum decomposition arises out of Ricci flow？



Consider $S^1 \times I$ (I is an interval)between two parts of a 2-manifolds。

$S^2 \times I$

期待 Ch 8：The details of how the pinching off (by surgery 手術)actually happens will in Chapter 8。

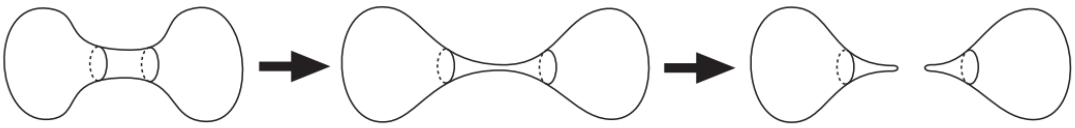


Figure 2.3: A neck "pinching off" in a 2-manifold. This diagram is intended to illustrate by lower-dimensional analogy what a neckpinch in a 3-manifold is like – the Ricci flow on 2-manifolds does not give rise to neckpinches.

The torus decomposition arises in a different way。

§ 05 Hamilton Ricci flow

$$\frac{\partial g}{\partial t} = (r - R(x,t))g(x,t) \ldots (*) \quad x \in M, t > 0$$

Theorem 1.1 (Hamilton). *Let* (M, g) *be a compact oriented Riemannian surface.*

(1) *If* M *is not diffeomorphic to the 2-sphere* S^2, *then any metric* g *converges to a constant curvature metric under equation* $(*)$.

(2) *If* M *is diffeomorphic to* S^2, *then any metric* g *with positive Gaussian curvature on* S^2 *converges to a metric of constant curvature under* $(*)$.

Theorem 1.2. *If* g *is any metric on* S^2, *then under Hamilton's Ricci flow, the Gaussian curvature becomes positive in finite time.*

Combining the two theorems above yields:

Corollary 1.3. *If* g *is any metric on a Riemann surface, then under Hamilton's Ricci flow,* g *converges to a metric of constant curvature.*

[The Ricci flow on the 2-sphere] Bennet Chow

1. An Illustrated introduction to the Ricci flow Gabriel Khan [部落格] 內有本書

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