Preliminaries from Riemannian geometry

§ 1-1 Riemannian metrics and Levi-Civita connection

For any vector bundle $\mathcal{V}$ over M we denote by $\Gamma(\mathcal{V})$ the vector space of smooth sections of $\mathcal{V}$.

Theorem 1.2 Levi-Civita connection

Given a Riemannian metric g on M，there uniquely exists a torsion-free connection on TM making g parallel，i.e.，there is a unique R-linear mapping

$\nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$ satisfying the Leibnitz formula

$\nabla(fX) = df \otimes X + f\nabla X$ ，and for all vector fields X and Y

1. $d(g(X,Y)) = g(\nabla X, Y) + g(X, \nabla Y)$
   (∇ compatible with g ， $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$)
2. $\nabla_X Y - \nabla_Y X - [X,Y] = 0$ ，torsion free

In local coordinates $(x^1,\ldots,x^n)$ the Levi-Civita connection ∇ is given by

$\nabla_{\partial_i}(\partial_j) = \Gamma_{ij}^k \partial_k$ ，where $\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l})$

The covariant derivative allows us to define the Hessian of a smooth function at any point, not just a critical point. Let f be a smooth real-valued function on M. We define the Hessian of f, denoted $\mathrm{Hess}(f)$, as follows:

$$\mathrm{Hess}(f)(X,Y) = X(Y(f)) - \nabla_X Y(f). \tag{1.2}$$

LEMMA 1.3. *The Hessian is a contravariant, symmetric two-tensor, i.e., for vector fields X and Y we have*

$$\mathrm{Hess}(f)(X,Y) = \mathrm{Hess}(f)(Y,X)$$

and

$$\mathrm{Hess}(f)(\phi X, \psi Y) = \phi\psi \mathrm{Hess}(f)(X,Y)$$

for all smooth functions ϕ, ψ. Other formulas for the Hessian are

$$\mathrm{Hess}(f)(X,Y) = \langle \nabla_X(\nabla f), Y \rangle = \nabla_X(\nabla_Y(f)) = \nabla^2 f(X,Y).$$

Also, in local coordinates we have

$$\mathrm{Hess}(f)_{ij} = \partial_i \partial_j f - (\partial_k f)\Gamma_{ij}^k.$$

The Laplacian Δf is defined as the trace of the Hessian。That is to say，in local coordinates near p we have $\Delta f(p) = \sum_{ij} g^{ij} Hess(f)(\partial_i, \partial_i)$ 。

Thus，if $\{X_i\}$ is an orthonormal basis for T_pM then $\Delta f(p) = \sum_i Hess(f)(X_i, X_i)$

§ 1-2 Cuvature of a Riemannian manifold

$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

In local coordinates the curvature tensor can be represented as

$R(\partial_i, \partial_j)\partial_k = R_{ijk}^l \partial_l$，where $R_{ijk}^l = \dfrac{\partial \Gamma_{jk}^l}{\partial x^i} - \dfrac{\partial \Gamma_{ik}^l}{\partial x^j} + \sum_m \Gamma_{jk}^m \Gamma_{im}^l - \sum_m \Gamma_{ik}^m \Gamma_{jm}^l$

$R(X,Y,Z,W) = g(R(X,Y)W,Z)$，$R(\partial_i, \partial_j, \partial_k, \partial_l) = R_{ijkl} = g_{km} R_{ijl}^m$

Claim 1.5. *The Riemann curvature tensor $\mathcal{R}$ satisfies the following properties:*

- *(Symmetry)* $R_{ijkl} = -R_{jikl}$, $R_{ijkl} = -R_{ijlk}$, $R_{ijkl} = R_{klij}$.
- *(1st Bianchi identity) The sum of R_{ijkl} over the cyclic permutation of any three indices vanishes.*
- *(2nd Bianchi identity)* $R_{ijkl,h} + R_{ijlh,k} + R_{ijhk,l} = 0$, *where*

$$R_{ijkl,h} = (\nabla_{\partial_h} \mathcal{R})_{ijkl}.$$

1st Bianchi identity： $\mathrm{R}_{jkl}^i + R_{klj}^i + R_{ljk}^i = 0$

2nd Bianchi identity： $R_{ijk,l}^h + R_{ikl,j}^h + R_{ilj,k}^h = 0$ L. Bianchi 1902

The sectional curvature of a 2-plane $P \subset T_pM$ is defined as

$K(P) = R(X,Y,X,Y)$，where {X,Y} is an orthonormal basis of P。

$K(P) = R_{ijkl} X^i Y^j X^k Y^l$，where $X = X^i \partial_i, Y = Y^i \partial_i$

$K(\pi) := \langle R(e_1, e_2)e_2, e_1 \rangle$ where $\{e_1, e_2\}$ is an orthonormal basis of π

Prove $K(\pi) = \dfrac{\langle R(X,Y)Y, X \rangle}{|X|^2 |Y|^2 - \langle X, Y \rangle^2}$

Ricci curvature tensor

$Ric = \sum_{i,j} R_{ij} dx^i \otimes dx^j$ where $R_{ij} = \sum_k R_{kij}^k$，$R_{ij} = \partial_k \Gamma_{ij}^k - \partial_j \Gamma_{ik}^k + \Gamma_{ij}^k \Gamma_{km}^m - \Gamma_{im}^k \Gamma_{jk}^m$

Scalar curvature

$R = tr_g Ric \quad S(p) = R := \sum_{i,j} g^{ij} R_{ij}$

§ 2.1 Consequences of Bianchi identities

§ 2.2 First examples

單值化定理

THEOREM 1.11. **(Uniformization Theorem)** *If (M^n, g) is a complete, simply-connected Riemannian manifold of constant sectional curvature λ, then:*

(1) *If $\lambda = 0$, then M^n is isometric to Euclidean n-space.*
(2) *If $\lambda > 0$ there is a diffeomorphism $\phi\colon M \to S^n$ such that $g = \lambda^{-1}\phi^*(g_{\mathrm{st}})$ where g_{st} is the usual metric on the unit sphere in $\mathbb{R}^{n+1}$.*
(3) *If $\lambda < 0$ there is a diffeomorphism $\phi\colon M \to \mathbb{H}^n$ such that $g = |\lambda|^{-1}\phi^*(g_{\mathrm{st}})$ where g_{st} is the Poincaré metric of constant curvature -1 on $\mathbb{H}^n$.*

Of course, if (M^n, g) is a complete manifold of constant sectional curvature, then its universal covering satisfies the hypothesis of the theorem and hence is one of $S^n, \mathbb{R}^n$, or $\mathbb{H}^n$, up to a constant scale factor. This implies that (M, g) is isometric to a quotient of one of these simply connected spaces of constant curvature by the free action of a discrete group of isometries. Such a Riemannian manifold is called a *space-form*.

The Uniformization Theorem can be stated in two forms. The first form emphasizes the universal covers and the second suggests the importance of the theorem by describing how the three universal covers give us information about every Riemann surface.

Theorem 2.1 (The Uniformization Theorem, version 1). *Up to biholomorphism, there are just three simply connected Riemann surfaces: the complex plane $\mathbb{C}$, the Riemann sphere $\hat{\mathbb{C}}$, and the open disk $\mathbb{D}$.*

Theorem 2.2 (The Uniformization Theorem, version 2). *Every connected Riemann surface X is biholomorphic to a quotient of one of $\hat{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{D}$ by the covering space action of a subgroup Γ of its automorphism (self-biholomorphism) group.*

Einstein manifold

Let M be n-dimensional manifold with n being either 2 or 3。If (M,g) is Einstein with Einstein constant λ，then M has constant sectional curvature $\frac{\lambda}{n-1}$，so that in fact M is a space-form。

§ 2.3 Cones

Let (N,g) be a Riemannian manifold。

Open cone over (N,g)： $N\times(0,\infty)$ with $\tilde{g}$

$\tilde{g}(x,s)=s^2g(x)+ds^2$ for any $(x,s)\in N\times(0,\infty)$

Fix local coordinates $(x^1,x^2,...,x^n)$ on N。Set $x^0=s$ ，in the local coordinates $(x^0,x^1,...,x^n)$ for the cone，the relation between Γ_{ij}^k and $\tilde{\Gamma}_{ij}^k$ are …

Denote by $\mathcal{R}_g$ the curvature tensor for g and by $\mathcal{R}_{\widetilde{g}}$ the curvature tensor for $\widetilde{g}$. Then the above formulas lead directly to:

$$\mathcal{R}_{\widetilde{g}}(\partial_i,\partial_j)(\partial_0)=0;\qquad 0\le i,j\le n,$$
$$\mathcal{R}_{\widetilde{g}}(\partial_i,\partial_j)(\partial_i)=\mathcal{R}_g(\partial_i,\partial_j)(\partial_i)+g_{ii}\partial_j-g_{ji}\partial_i\qquad 1\le i,j\le n.$$

後面有一個命題 一個推論 略

Geodesics and the exponential map

§ 3.1 Geodesics and the energy functional

Let I be an open interval。γ:I → M is a smooth curve。

γ is called a geodesic if $\nabla_T T=0$ ，where $T=\dfrac{d\gamma}{dt}$ 。

In local coordinates, we write $\gamma(t)=(x^1(t),\ldots,x^n(t))$ and this equation becomes

$$0=\nabla_{\dot\gamma}\dot\gamma(t)=\left(\sum_k\left(\ddot{x}^k(t)+\dot{x}^i(t)\dot{x}^j(t)\Gamma_{ij}^k(\gamma(t))\right)\partial_k\right).$$

So the geodesic equation is $\ddot{x}^k+\Gamma_{ij}^k\dot{x}^i\dot{x}^j=0$

Hopf-Rinow

If (M,g) is complete as a metric space，then every geodesic extends to a geodesic defined for all time。

Geodesics are critical points of the energy functional。

§ 3.2 Families of geodesics and Jacobi fields

Jacobi equation

$\nabla_X\nabla_X Y+R(Y,X)X=0$

A vector field Y along a geodesic γ is said to be a Jacobi field if it satifies this equation and vanishes at the initial point p。

Jacobi fields are also determined by the energy functional。

§ 3.3 Minimal geodesics

Conjugate point

Let γ be a geodesic beginning at $p \in M$ 。For any t>0 we say that $q = \gamma(t)$ is a conjugate point along γ if there is a non0zero Jacobu field along γ vanishing at $\gamma(t)$ 。

PROPOSITION 1.20. *Suppose that $\gamma\colon [0,1] \to M$ is a minimal geodesic. Then for any $t < 1$ the restriction of γ to $[0,t]$ is the unique minimal geodesic between its endpoints and there are no conjugate points on $\gamma([0,1))$, i.e., there is no non-zero Jacobi field along γ vanishing at any $t \in [0,1)$.*

§ 3.4 The exponential mapping

$\gamma_v : [0,1] \to M$ is a geodesic starting from p with initial velocity vector v。

$\exp_p(v) = \gamma_v(1)$

By the Hopf-Rinow theorem，if M is complete，then the exponential map is defined on all of T_pM 。

後面有些細節尚待釐清。

§ 4 Computations in Gaussian normal coordinates

§ 5 Basic curvature comparison results

THEOREM 1.31. *(Sectional Curvature Comparison) Fix $k \geq 0$. Let (M,g) be a Riemannian manifold with the property that $-k \leq K(P)$ for every 2-plane P in TM. Fix a minimizing geodesic $\gamma\colon [0,r_0) \to M$ parameterized at unit speed with $\gamma(0) = p$. Impose Gaussian polar coordinates $(r,\theta^1,\ldots,\theta^{n-1})$ on a neighborhood of γ so that $g = dr^2 + g_{ij}\theta^i \otimes \theta^j$. Then for all $0 < r < r_0$ we have*

$$(g_{ij}(r,\theta))_{1\leq i,j\leq n-1} \leq \mathrm{sn}_k^2(r),$$

and the shape operator associated to the distance function from p, f, satisfies

$$(S_{ij}(r,\theta))_{1\leq i,j\leq n-1} \leq \sqrt{k}\,\mathrm{ct}_k(r).$$

There is also an analogous result for a positive upper bound to the sectional curvature, but in fact all we shall need is the local diffeomorphism property of the exponential mapping.

LEMMA 1.32. *Fix $K \geq 0$. If $|\mathrm{Rm}(x)| \leq K$ for all $x \in B(p, \pi/\sqrt{K})$, then $\exp_p$ is a local diffeomorphism from the ball $B(0, \pi/\sqrt{K})$ in T_pM to the ball $B(p, \pi/\sqrt{K})$ in M.*

There is a crucial comparison result for volume which involves the Ricci curvature.

THEOREM 1.33. **(Ricci curvature comparison)** *Fix $k \geq 0$. Assume that (M, g) satisfies $\mathrm{Ric} \geq -(n-1)k$. Let $\gamma \colon [0, r_0) \to M$ be a minimal geodesic of unit speed. Then for any $r < r_0$ at $\gamma(r)$ we have*

$$\sqrt{\det g(r,\theta)} \leq \mathrm{sn}_k^{n-1}(r)$$

and

$$\mathrm{Tr}(S)(r,\theta) \leq (n-1)\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}.$$

THEOREM 1.34. *(Relative Volume Comparison, Bishop-Gromov 1964-1980) Suppose (M, g) is a Riemannian manifold. Fix a point $p \in M$, and suppose that $B(p, R)$ has compact closure in M. Suppose that for some $k \geq 0$ we have $\mathrm{Ric} \geq -(n-1)k$ on $B(p, R)$. Recall that H_k^n is the simply connected, complete manifold of constant curvature $-k$ and $q_k \in H_k^n$ is a point. Then*

$$\frac{\mathrm{Vol}\, B(p,r)}{\mathrm{Vol}\, B_{H_k^n} B(q_k, r)}$$

is a non-increasing function of r for $r < R$, whose limit as $r \to 0$ is 1. In particular, if the Ricci curvature of (M, g) is ≥ 0 on $B(p, R)$, then $\mathrm{Vol}\, B(p,r)/r^n$ is a non-increasing function of r for $r < R$.

§ 6. Local volume and the injectivity radius

PROPOSITION 1.35. *Fix an integer $n > 0$. For every $\epsilon > 0$ there is $\delta > 0$ depending on n and ϵ such that the following holds. Suppose that (M^n, g) is a complete Riemannian manifold of dimension n and that $p \in M$. Suppose that $|\mathrm{Rm}(x)| \leq r^{-2}$ for all $x \in B(p, r)$. If the injectivity radius of M at p is at least ϵr, then $\mathrm{Vol}(B(p, r)) \geq \delta r^n$.*

THEOREM 1.36. *Fix an integer $n > 0$. For every $\epsilon > 0$ there is $\delta > 0$ depending on n and ϵ such that the following holds. Suppose that (M^n, g) is a complete Riemannian manifold of dimension n and that $p \in M$. Suppose that $|\mathrm{Rm}(x)| \leq r^{-2}$ for all $x \in B(p, r)$. If $\mathrm{Vol}\, B(p, r) \geq \epsilon r^n$ then the injectivity radius of M at p is at least δr.*