H.Poincare1904 R.Hamilton1982 G.Perelman2006

§ Introduction

Poincare conjecture：

A closed，smooth，simply connected 3-manifold is diffeomorphic to S^3 。

The difficulty was to deal with singularities in the Ricci flow。

The uniformization theorem(單值化定理)：

黎曼面單值化定理是曲面微分幾何最為深刻而基本的定理之一，其證明方法豐富多彩，例如基於複分析的古典方法，基於 Ricci 流的現代方法，基於射影結構的代數方法等等。

定理 （Poincare-Koebe Uniformization）任意一個單聯通的黎曼面都和三個標準黎曼面中的一個共形等價：

擴展複平面 $C \cup \{\infty\}$ ，S^2 ，複平面 C 或者單位圓盤 D

The Geometrization conjecture：William Thurston 1970

Thurston's eight geometries：

H^3 $H^2 \times R$ R^3 $S^2 \times R$ S^3 Sol $\widetilde{SL}_2$ Nil

With these geometries, he proposed a classification for the possible geometries of three-dimensional spaces similar to how the uniformization theorem classifies the geometry of two-dimensional surfaces.

Stephen Smale 1962

Michael Freedman 1982

John Milnor

3-dimensional spherical space-form conjecture：

Any closed 3-manifold with finite fundamental group is diffeomorphic a 3-dimensinoa spherical space-form。

Theorem

Let M be a closed，connected 3-manifold and suppose that the fundamental group of M is a free product of finite groups and infinite cycle groups。Then M is diffeomorphic to a connected sum of spherical space-forms，copies of $S^2 \times S^1$，and copies of the unique (up to diffeomorphism) non-orientable 2-sphere bundle over S^1。

§ Ricci flow

On harmonic maps：heat flow

James Eells

Joseph Sampson

R. Hamilton 1982

考慮在 $M^n \times [0,T]$ 上的 Ricci flow $\dfrac{\partial g}{\partial t} = -2Ric(g(t))$ …(1)

The Ricci flow is a degenerate parabolic evolution system on metric。

及 Normalized Ricci flow $\dfrac{\partial g}{\partial t} = -2Ric(g) + \dfrac{2}{n} rg$ …(2)

其中 $r = \dfrac{\int Rdn}{d\mu}$ 為平均純曲率，R 純曲率。

在封閉的 3 維流形 M^3 上，若(2)初始值度量 $g_{ij}(0)$ 里奇曲率正的，則對所有時間 t，(2)的解存在，$g_{ij}(t)$ 逼近到正的常曲率度量，且 M^3 與 $\dfrac{S^3}{\Gamma}$ 可微同胚，其中 Γ 為有限群。

Hamilton used the Nash-Moser implicit function theorem to prove the following short-time existence and uniqueness theorem for the Ricci flow on compact manifold。

Let (M, g) be a compact Riemannian manifold。Then there exists a constant T>0 such that the Ricci flow $\dfrac{\partial g}{\partial t} = -2Ric(g)$，with $g_{ij}(x,0) = g_{ij}(x)$，admits a unique smooth solution $g_{ij}(x,t)$ for all $x \in M$ and $t \in [0,T)$

R. Hamilton 的 Ricci flow 基本上是利用幾何的方法來研究 William Thurston's Geometrization conjecture。

大致上分為兩部份

1. 研究有限時間內(1)的奇異解
2. 非奇異解的分類

Hamilton showed that starting with a simply connected three-dimensional space whose Ricci curvature is positive，it would indeed evolve to a round sphere。

This was a weaker version of the Poincare conjecture，but a major proof of concept for the power of the Ricci flow。

Furthermore，he was nearly able to find a new proof of the uniformization theorem using Ricci ow，and this proof was completed by Bennett Chow and Xiuxiong Chen，Peng Lu，and Gang Tian。

These results suggested that the Ricci flow was a promising approach to the Poincare conjecture，but there was a problem。

The flow would encounter singularities，which are times where the space would either collapse to a point15 or violently rip itself apart。

Hamilton understood the cases when the space would shrink to a point，but was not able to control the geometry in the latter case，and this presented a fundamental obstacle to finishing the proof。

Grigori Perelman

例 If $Ric(g_0)=\lambda g_0$，λ is a constant。Then a solution g(t) of $\frac{\partial g}{\partial t}=-2Ric(g)$ with $g(0)=g_0$ is given by $g(t)=(1-2\lambda t)g_0$

In particularly， for (S^n, g_0)，we have $Ric(g_0)=(n-1)g_0$，so the evolution is

$g(t)=(1-2(n-1)t)g_0$ and the sphere collapses to a point at time $T=\frac{1}{2(n-1)}$

例 A hyperbolic metric g_0，the sectional curvature =-1。

In this case，$Ric(g_0)=-(n-1)g_0$，the evolution is g(t)=(1+2(n-1)t) g_0 and the manifold expands homothetically (相似 同位)for all time。

§ Ricci solitons(孤立子)

A complete Riemannian manifold (M ,g) is called a Ricci soliton，if and only if，there exists a smooth vector field V such that $Ric(g)=\lambda g-\frac{1}{2}L_V g$ for some constant λ。

If there exists a function $f:M\to R$ such that $V=\nabla f$，we call (M ,g) a gradient Ricci soliton and the soliton equation becomes $Ric(g)+\nabla^2 f=\lambda g$

§ Hamilton cigar soliton

Let $M=R^2$，$g_0=\rho^2(dx^2+dy^2)$

The Gauss curvature $K=-\frac{1}{\rho^2}\Delta\ln\rho$，$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$

Then $Ric(g_0)=Kg_0$，if we set $\rho^2=\frac{1}{1+x^2+y^2}$，we will find $K=\frac{2}{1+x^2+y^2}$

That is $Ric(g_0)=\frac{2}{1+x^2+y^2}g_0$，meanwhile，if we define $Y:=-2(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y})$

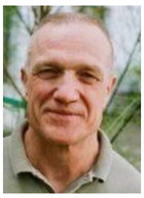
Then $L_Y g_0=-\frac{4}{1+x^2+y^2}g_0$，by (1.2.4) $-2Ric(g_0)=L_Y g_0-2\lambda g_0$

$\lambda=0$，g_0 is a steady Ricci flow。

If write g_0 in terms of the geodesic distance from the origin，and polar angle to give $g_0=ds^2+\tanh^2 s d\theta^2$

This show that the cigar opens at infinity like a cylinder，and therefore looks like a cigar！The curvature in these coordinates is $K=\frac{2}{\cosh^2 s}$

Finally，note that the cigar is also a gradient soliton since Y is radial。Indeed we may take $f=-2\ln\cosh s$。



§ The Bryant soliton

Robert L. Bryant https://www.msri.org/people/staff/bryant/

[3DXM Surface Gallery]

[The Modeling of Degenerate Neck Pinch Singularities in Ricci Flow by Bryant Solitons]

§ The Gaussian soliton

Let X(t) be a time dependent family of smooth vector fields on M(not necessary compact) ，generating a family of diffeomorphisms ψ_t 。

$X(\psi_t(y),t)f = \frac{\partial f \circ \psi_t}{\partial t}(y)$ for a smooth function $f: M \to R$