§ Curvature of a Riemannian manifold

The Riemann curvature tensor of M is the (1,3)-tensor on M

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

In local coordinates， $R(\partial_i, \partial_j)\partial_k = \mathrm{R}_{ijk}^{l} \partial_l$

Where $R_{ijk}^{l} = \partial_i \Gamma_{jk}^{l} - \partial_j \Gamma_{ik}^{l} + \Gamma_{jk}^{m} \Gamma_{im}^{j} - \Gamma_{ik}^{m} \Gamma_{im}^{l}$

$R(X,Y,Z,W) = g(R(X,Y)W, Z)$ is a (0,4)-tensor

In local coordinates

$$R(\partial_i, \partial_j, \partial_k, \partial_l) = R_{ijkl} = g_{km} R_{ijl}^{m}$$

Claim 1.5. *The Riemann curvature tensor $\mathcal{R}$ satisfies the following properties:*

- *(Symmetry)* $R_{ijkl} = -R_{jikl}$, $R_{ijkl} = -R_{ijlk}$, $R_{ijkl} = R_{klij}$.
- *(1st Bianchi identity) The sum of R_{ijkl} over the cyclic permutation of any three indices vanishes.*
- *(2nd Bianchi identity)* $R_{ijkl,h} + R_{ijlh,k} + R_{ijhk,l} = 0$, *where*

$$R_{ijkl,h} = (\nabla_{\partial_h} \mathcal{R})_{ijkl}.$$

§ sectional curvature of a 2-plane $P \subset T_p M$

K(P)=R(X,Y,X,Y)，where {X,Y} is an orthonormal basis of P。

In local coordinates，suppose that $X = X^i \partial_i, Y = Y^i \partial_i$ ，then

$$K(P) = R_{ijkl} X^i Y^j X^k Y^l$$

S^{n-1} the sphere of radius r in R^n has constant sectional curvature $\frac{1}{r^2}$ 。

R^n with the Euclidean metric has constant sectional curvature 0

Hyperbolic space H^n has constant setional curvature -1

§ Ricci curvature tensor

$Ric(X,Y) = g^{kl} R(X, \partial_k, \mathrm{Y}, \partial_l)$ in local coordinates

§ scalar curvature $R = tr_g Ric = g^{ij} Ric_{ij}$

§ Consequences of the Bianchi identities

$\nabla_\sigma R_{\alpha\beta\mu\nu} + \nabla_\upsilon R_{\alpha\beta\sigma\mu} + \nabla_\mu R_{\alpha\beta\upsilon\sigma} = 0$ will be of fundamental important to find the Einstein equation。

For any contravariant two-tensor ω on M (such as Ric or Hess(f))，we define the contravariant one-tensor $div(\omega)$

$$div(\omega)(X) = \nabla^* \omega(X) = g^{rs} \nabla_r (\omega)(X, \partial_s)$$

Then $dR = 2div(Ric) = 2\nabla^* Ric$

§ Uniformization Theorem

If (M,g) is a complete，simply-connected Riemannian mnifold of constant sectional curvature λ，then

(1) If $\lambda = 0$，then M is isometric to Euclidean n-space

(2) If $\lambda > 0$，there is a diffeomorphism $\phi : M \to S^n$ such that $g = \lambda^{-1} \phi^* (g_{ij})$，

where g_{ij} is the usual metric on the unit sphere in R^{n+1}

(3) If $\lambda < 0$，there is a diffeomorphism $\phi : M \to H^n$ such that $g = |\lambda|^{-1} \phi^* (g_{ij})$，

where g_{ij} is the Poincare metric of constant curvature -1 on H^n。

Of course, if (M^n, g) is a complete manifold of constant sectional curvature, then its universal covering satisfies the hypothesis of the theorem and hence is one of $S^n, \mathbb{R}^n$, or $\mathbb{H}^n$, up to a constant scale factor. This implies that (M, g) is isometric to a quotient of one of these simply connected spaces of constant curvature by the free action of a discrete group of isometries. Such a Riemannian manifold is called a *space-form*.

§ Einstein manifold

$Ric(g) = \lambda g$

EXAMPLE 1.13. Let M be an n-dimensional manifold with n being either 2 or 3. If (M, g) is Einstein with Einstein constant λ, one can easily show that M has constant sectional curvature $\frac{\lambda}{n-1}$, so that in fact M is a space-form.