§ Hessian of a smooth function

$Hess(f)(X,Y) = X(Y(f)) - \nabla_X Y(f)$

Lemma

The Hessian is a contravariant，symmetric two-tensor。

For any vector fields X and Y

1. Hess(f)(X,Y)=Hess(f)(Y,X)。The proof of symmetry is direct from the torsion-free assumption。

   $Hess(f)(X,Y) - Hess(f)(Y,X) = [X,Y](f) - (\nabla_X Y - \nabla_Y X)(f) = 0$

   對稱 $\nabla_X Y - \nabla_Y X = [X,Y]$(called torsion free)

2. $Hess(f)(\phi X, \psi Y) = \phi\psi Hess(f)(X,Y)$ for all smooth functions ϕ, ψ

Other formulas for the Hessian are

1. $Hess(f)(X,Y) = \langle \nabla_X(\nabla f), Y \rangle = \nabla_X(\nabla_Y(f)) = \nabla^2 f(X,Y)$

   $\langle \nabla_X(\nabla f), Y \rangle = X(\langle \nabla f, Y \rangle - \langle \nabla f, \nabla_X Y \rangle = X(Y(f)) - \nabla_X Y(f) = Hess(f)(X,Y)$

2. $Hess(f)_{ij} = \partial_i \partial_j f - (\partial_k f)\Gamma_{ij}^k$ in local coordinates

$$\nabla^2 u = \begin{pmatrix} u_{x_1x_1} & u_{x_1x_2} & \cdots & u_{x_1x_n} \\ u_{x_2x_1} & u_{x_2x_2} & \cdots & u_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{x_nx_1} & u_{x_nx_2} & \cdots & u_{x_nx_n} \end{pmatrix}.$$ is called the Hessian matrix of u

The Laplacian Δf is defined as the trace of the Hessain：

In local coordinates near p，we have $\Delta f(p) = \sum_{ij} g^{ij} Hess(f)(\partial_i, \partial_j)$

Thus，if $\{X_i\}$ is an orthonormal basis for T_pM then $\Delta f(p) = \sum_i Hess(f)(X_i, X_i)$

Since $df = (\partial_r f)dx^r$ and $\nabla(dx^k) = -\Gamma_{ij}^k dx^i \otimes dx^j$，it follows that

$\nabla(df) = (\partial_i \partial_j f - (\partial_k f)\Gamma_{ij}^k) dx^i \otimes dx^j$，It is direct from the definition that

$Hess(f)_{ij} = Hess(f)(\partial_i, \partial_j) = \partial_i \partial_j f - (\partial_k f)\Gamma_{ij}^k$

§ 參考[Extrema]