§ Curve-shortening flow

§ steepest descent flow for length

Given a smooth immersion $X_0 : R/Z \to R^2$

We can evolve it by taking $X : R/Z \times [0,T) \to R^2$ such that

$$\begin{cases} \dfrac{\partial X}{\partial t}(u,t) = -\kappa N(u,t) \\ X(u,0) = X_0(u) \end{cases}$$ …This is the CSF (curve-shortening flow) equation。

Here κ is the curvature of our curve，N is the unit normal vector，defined by

$$-\kappa N = \frac{\partial^2 X}{\partial s^2}$$

Example: The shrinking circle. If our initial curve X_0 is a circle of radius r_0 centred at the origin, then the solution will take the form $X(u,t) = r(t)(\cos(u), \sin(u))$. We have $\mathbf{N} = X/r, \kappa = 1/r$, so the CSF equation becomes

$$\frac{dr}{dt} = -\frac{1}{r}$$

which has the solution

$$X(u,t) = (r_0^2 - 2t)^{\frac{1}{2}}(\cos(u), \sin(u)). \tag{4.2}$$

So the circle shrinks to a point at a finite time $t = \sqrt{r_0^2/2} = r_0/\sqrt{2}$.

PDE103heatequation p.6 example

$$\begin{cases} u_t = k u_{xx}, t > 0 \\ u\big|_{t=0} = x \end{cases}$$ the solution is $u(x,t) = \dfrac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{4kt}} y dy = \ldots = x$

$$\begin{cases} \dfrac{\partial X}{\partial t}(u,t) = \dfrac{\partial^2 X}{\partial s^2} \\ X(u,0) = X_0(u) \to r_0(\cos u, \sin u) \end{cases} \quad \ldots(*)$$

Then the solution of (*) is $X(u,t) = r(t)(\cos u, \sin u)$ where $r(0) = r_0$

§ short time existence

§ finite time singularity

Theorem 4.3. ***The Avoidance Principle.*** *If X, Y are solutions of the CSF on $[0,T)$, and $X(u,0) \neq Y(v,0)$ for all u, v, then $X(u,t) \neq Y(v,t)$ for all u, v, for all $t \in [0,T)$. That is, if the curves do not intersect initially, they will not intersect at any later time.*

This theorem can be proven using a version of the maximum principle to show that the smallest distance between the two curves is a nondecreasing function of time.

Theorem 4.4. *The CSF exists for only a finite time before becoming singular. That is, the maximal time T mentioned in Theorem 4.2 is finite.*

§ curvature explodes

Theorem 4.5. *Suppose we have a solution of the* CSF *on a* ***maximal*** *time interval* $[0, T)$*, where* $T < \infty$ *by Theorem 4.4. Then*

$$\lim_{t \to T} \sup\{|\kappa(u,t)| : u \in \mathbb{R}/\mathbb{Z}\} = \infty.$$

§ Grayson theorem

Theorem 4.6. *(Grayson) Given any embedded circle* X_0*, the solution to the* CSF $X(u,t)$ *with* $X(u,0) = X_0(u)$ *will remain embedded and will shrink to a point* x *as* t *approaches the maximal time of existence* T*. Furthermore, if* $A(t)$ *denotes the area enclosed by the curve* $X(\cdot, t)$ *at time* t*, then the curve*

$$\tilde{X}(u,t) = \sqrt{\frac{\pi}{A(t)}}(X(u,t) - x)$$

converges exponentially to a unit circle as $t \to T$*.*