The KdV equation：

§ 01　a wave solution

$$\begin{cases} u_t + 6uu_x + u_{xxx} = 0 \\ u(x,0) = f(x) \end{cases}, -\infty < x < \infty, 0 \le t < \infty$$

Is used to describe the evolution of shallow water wave。

A traveling wave solution which has permanent form occurs to a balance of its dispersive (相散)term u_{xxx}，and its nonlinear term $6uu_x$。

Let $\xi = x - ct$

Let $u(x,t) = f(x-ct)$，$u_t = \frac{df}{d\xi}\frac{d\xi}{dt} = -c\frac{df}{d\xi}$ then $-cf' + 6ff' + f''' = 0 ...(1)$

(1) 積分一次得 $-cf + 3f^2 + f'' = A$ 將 f'視為積分因子(即兩邊同乘以 f')

$$f'f'' = Af' + cff' - 3f^2 f'$$

$$[\frac{1}{2}(f')^2]' = (Af)' + (\frac{c}{2}f^2)' - (f^3)'$$ 再積分得

$$(f')^2 = 2Af + cf^2 - 2f^3 + B$$

考慮邊界值 $f, f', f'' \to 0$ as $x \to \infty$ then A=B=0

$(f')^2 = cf^2 - 2f^3 = f^2(c-2f)$，其中 $c - 2f > 0$

$\int \frac{df}{f(c-2f)^{\frac{1}{2}}} = \pm\int d\xi$，let $f = \frac{c}{2}\text{sech}^2\theta$ then $c - 2f = ... = c\tanh^2\theta$

其中

$\cosh x = \frac{e^x + e^{-x}}{2}, \sinh x = \frac{e^x - e^{-x}}{2}, \cosh^2 x - \sinh^2 x = 1$，$\frac{d}{dx}\text{sech}\, x = -\tanh x\, \text{sech}\, x$

$f' = \frac{df}{d\xi}, \xi = x - ct$，$df = c\,\text{sech}\,\theta(-\tanh\theta\,\text{sech}\,\theta)d\theta$

$\int \frac{df}{f(c-2f)^{\frac{1}{2}}} = ... = \frac{-2\theta}{\sqrt{c}} = \pm(\xi + k) = \pm(\text{x-ct+x}_0)$，$\theta = \frac{\sqrt{c}}{2}(x - ct + x_0)$

$$u(x,t) = f(x-ct) = \frac{c}{2}\text{sech}^2[\frac{\sqrt{c}(x-ct+x_0)}{2}]$$

摘自[孤立波淺談]