Soliton Solutions to the Korteweg-de Vries Equation

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Abstract

The Korteweg-de Vries (KdV) equation, a nonlinear partial differential equation, models waves in shallow water. My study explores soliton solutions to the KdV equation using the scattering transform. Example initial value problems are given to illustrate this process.

Contents

1 Introduction to the Korteweg-de Vries (KdV) Equation

The Korteweg-de Vries (KdV) equation is a partial differential equation that describes certain types of wave phenomena. It is given by:

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1.1}$$

It was independently discovered by Boussinesq (1877) and by Diederik Korteweg and Gustav de Vries (1895). The KdV equation has applications in modeling waves in shallow water. Our objective is to obtain soliton solutions to the KdV equation. Our main reference is [3].

1.1 Solitary Waves

Solitary waves are a fascinating phenomenon that was first observed by John Scott Russell, a naval architect, in 1834 [1]. While observing a canal barge hitting an underwater obstruction, Russell noted the emergence of a unique wave pattern:

- A "bell-shaped crest" that emerged from the disturbance.
- This crest had a consistent height, independent of the cross-channel direction.
- The crest was characterized by its unchanging and steadily propagating nature.

Russell's observation is best captured in his own words:

> It accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth, and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed.
> — J. Scott Russell (1834)

1.2 A "Guess and Check" Solution

1. We start by making a guess about the form of the solution:

$$u(x,t) = f(X) = f(x - ct)$$

   where c represents the wave speed.

2. Substituting this guess into the KdV equation:

$$\begin{aligned}
0 &= -cf' - 6ff' + f''' \\
f''' &= cf' + 6ff' = cf' + 3(f^2)' \\
f'' &= cf + 3f^2 + A \\
f'f'' &= cff' + 3f^2f' + Af' \\
\frac{1}{2}[(f')^2]' &= \frac{c}{2}(f^2)' + (f^3)' + Af' \\
\frac{1}{2}[(f')^2] &= \frac{c}{2}f^2 + f^3 + Af + B \\
&\text{Assuming } f, f', f'' \to 0 \text{ as } X \to \pm\infty, \text{ we have } A, B = 0. \\
\frac{1}{2}[(f')^2] &= f^3 + \frac{c}{2}f^2 = f^2\left(f + \frac{c}{2}\right)
\end{aligned}$$

3. Solving for f gives:

$$f(X) = -\frac{c}{2}\operatorname{sech}^2(\theta) = -\frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(X - X_0)\right) \tag{1.2}$$

   where $\theta = \frac{\sqrt{c}}{2}(X - X_0)$.

1.3 Conservation Laws

Given $u(x,t), T = f(u), X = g(u)$, and u satisfies the following equation:

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0$$

This is a conservation law with density T and flux X. We have:

$$\frac{d}{dt}\int_{-\infty}^{\infty} T\,dx = -X\Big|_{-\infty}^{\infty} = 0,$$

when $X \to 0$ as $x \to \pm\infty$. Therefore, the integral of T over the entire domain is constant.

In the context of the KdV equation, one can find such conservation laws as follows.

1. Example 1:

$$\begin{aligned} 0 = u_t - 6uu_x + u_{xxx} &= \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(-3u^2 + u_{xx}) \\ \text{Take } T = u \text{ and } X &= u_{xx} - 3u^2 \\ \text{So } \int_{-\infty}^{\infty} u\,dx &= \text{constant} \end{aligned}$$

2. Example 2:

$$\begin{aligned} 0 = u(u_t - 6uu_x + u_{xxx}) &= \frac{\partial}{\partial t}(\frac{1}{2}u^2) + \frac{\partial}{\partial x}(-2u^3 + uu_{xx} - \frac{1}{2}u_x^2) \\ \text{Take } T = \frac{1}{2}u^2 \text{ and } X &= -2u^3 + uu_{xx} - \frac{1}{2}u_x^2 \\ \text{So } \int_{-\infty}^{\infty} u^2\,dx &= \text{constant} \end{aligned}$$

The KdV equation actually has infinitely many conservation laws, which can be generated systematically as below [4]. We define w such that

$$u = w + \epsilon w_x + \epsilon^2 w^2,$$

where u satisfies the KdV equation and ϵ is any real number. We can then obtain Gardner's equation

$$w_t - 6(w + \epsilon^2 w^2)w_x + w_{xxx} = 0$$

which has a conservation law

$$w_t + (-3w^2 - 2\epsilon^2 w^3 + w_{xx})_x = 0 \tag{1.3}$$

Suppose we can write w as:

$$w = \sum_{n=0}^{\infty} \epsilon^n w_n(u)$$

We can equate coefficients of ϵ^n to obtain conservation laws of the KdV Equation

$$\epsilon^0 w_0 + \epsilon^1 w_1 + \epsilon^2 w_2 + ... = u - \epsilon^1(\epsilon^0 w_{0x} + \epsilon^1 w_{1x} + \epsilon^2 w_{2x} + ...) - \epsilon^2(\epsilon^0 w_0 + \epsilon^1 w_1 + \epsilon^2 w_2 + ...)^2$$

For example, we obtain the following w_n

$$\begin{aligned} &\epsilon^0 : w_0 = u \\ &\epsilon^1 : w_1 = -w_{0x} = -u_x \\ &\epsilon^2 : w_2 = -w_{1x} - w_0^2 = u_{xx} - u^2 \end{aligned}$$

and we plug them into (1.3) to obtain conservation laws for the KdV Equation

$$\begin{aligned} &\epsilon^0 : u_t + (-3u^2 + u_{xx})_x = 0 \\ &\epsilon^1 : (-u_x)_t + (6uu_x - u_{xxx})_x \\ &\epsilon^2 : (u_{xx} - u^2)_t + (4u^3 - 8uu_{xx} - 5u_x^2 + u_{xxxx})_x = 0 \end{aligned}$$

By repeating the process of equating coefficients, we can generate conservation laws for $n = 0, 1, 2, 3, ...$

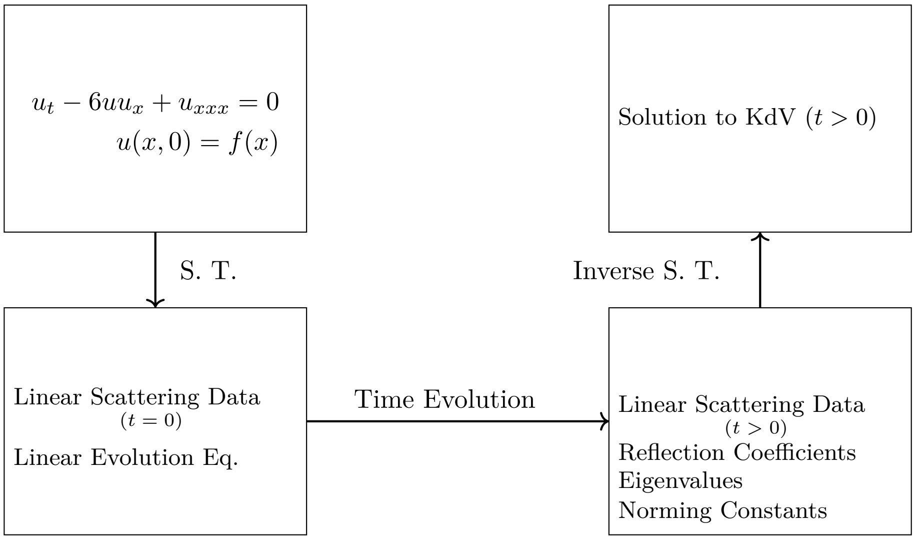

Figure 1: Idea of Scattering and Inverse Scattering

2 Initial Value Problem (IVP) for the KdV Equation

2.1 Idea of Scattering and Inverse Scattering

In Figure 1, we present the idea of the scattering transform and inverse scattering transform, which is a process similar to the Fourier Transform. We begin by considering the KdV IVP and applying the scattering transform to derive the scattering data, specifically the reflection coefficients, eigenvalues, and norming constants, at $t = 0$. These scattering data exhibit a simple time evolution, enabling us to determine them for $t > 0$. Finally, we utilize the inverse scattering transform to construct a solution for the KdV IVP based on the scattering data.

2.2 Steps of Scattering and Inverse Scattering

2.2.1 The Initial Value Problem

Consider the KdV equation and the initial condition:

$$u_t - 6uu_x + u_{xxx} = 0, \quad u(x,0) = g(x). \tag{2.1}$$

We first make some assumptions:

$$\int_{-\infty}^{\infty} \left| \frac{d^n g}{dx^n} \right|^2 < \infty \quad \text{for} \quad n = 0, 1, 2, 3, 4,$$
$$\int_{-\infty}^{\infty} (1 + |x|)|g(x)|\, dx < \infty.$$

These assumptions are made to ensure the existence of:

1. A unique smooth solution of the IVP (2.1).
2. A solution to an eigenvalue problem, which will be mentioned shortly.

2.2.2 The Scattering Problem

Consider $\psi(x,t), \lambda(t)$ such that:

$$\psi_{xx} + (\lambda - u)\psi = 0, \quad \psi < \infty \text{ as } x \to \pm\infty. \tag{2.2}$$

We refer to ψ as eigenfunctions and λ as eigenvalues. But why? Note that equation (2.2) can be transformed into $\psi_{xx} - u\psi = -\lambda\psi$, then we have $L\psi = \widetilde{\lambda}\psi$, where $L = \frac{\partial^2}{\partial x^2} - u$. We can then obtain two types of eigenfunctions:

1. Bound States ($\lambda < 0$)
   Let's define $k = \sqrt{-\lambda} > 0$. ψ will be bounded at infinity for only a finite number of discrete eigenvalues $k_n = \sqrt{-\lambda_n}$.

$$\psi_n(x) \sim \begin{cases} e^{k_n x} & \text{as } x \to -\infty \\ e^{-k_n x} & \text{as } x \to \infty \end{cases}$$

2. Unbound States ($\lambda > 0$)
   Let's define $k = \sqrt{\lambda} > 0$. In this case, ψ will exhibit sinusoidal behavior at infinity, ensuring automatic boundedness.

$$\psi(x,t) \sim \begin{cases} e^{-ikx} + b(k,t)e^{ikx} & \text{as } x \to \infty \\ a(k,t)e^{-ikx} & \text{as } x \to -\infty \end{cases} \tag{2.3}$$

   We refer to b as reflection coefficients and a as transmission coefficients.

2.2.3 Time Dependence of Scattering Data

First we discuss the λ time dependence. Define

$$R = \psi_t + u_x\psi - 2(u + 2\lambda)\psi_x. \tag{2.4}$$

Then we can see that

$$\psi R_x - \psi_x R = \psi\psi_{xt} - \psi_x\psi_t + \psi^2 u_{xx} - 2u_x\psi\psi_x - 2(u + 2\lambda)(\psi\psi_{xx} - \psi_x^2).$$

Then, differentiation of both sides yields

$$\frac{\partial}{\partial x}(\psi R_x - \psi_x R) = \psi^2(u_t - \lambda_t + u_{xxx} - 6uu_x).$$

Since $u(x,t)$ solves KdV (1.1), we have

$$\psi^2\lambda_t + \frac{\partial}{\partial x}(\psi R_x - \psi_x R) = 0.$$

Integrating both sides, we see that

$$\lambda_t \int_{-\infty}^{\infty} \psi^2\, dx = -[\psi R_x - \psi_x R]\Big|_{-\infty}^{\infty} = 0.$$

Since $\int_{-\infty}^{\infty} \psi^2\, dx$ is a nonzero constant, we conclude

$$\lambda_t = 0. \tag{2.5}$$

This has shown that eigenvalues are independent of time!

2.2.4 The Evolution Equation

We start with the equation

$$\frac{\partial}{\partial x}(\psi R_x - \psi_x R) = 0.$$

First, we consider the expression $\psi R_x - \psi_x R$. We notice that this quantity is a function of t only, which means it does not vary with respect to spatial coordinate x. Thus, we can find its quantity by taking x at infinity

$$\psi R_x - \psi_x R = \lim_{x\to\infty} (\psi R_x - \psi_x R) = 0.$$

Next, by quotient rule, we introduce the quantity $\frac{R}{\psi}$, and we observe that its spatial derivative is then zero:

$$\psi^2 \left(\frac{R}{\psi}\right)_x = 0.$$

This implies that $\frac{R}{\psi}$ is a function of t only, and we can obtain its value by taking x at infinity

$$\frac{R}{\psi} = \lim_{x\to\infty} \left(\frac{R}{\psi}\right) = \left[\frac{\psi_t + u_x\psi - 2(u+2\lambda)\psi_x}{\psi}\right]_{x\to\infty} = \frac{4\lambda k e^{-kx}}{e^{-kx}} = 4\lambda k = -4k^3.$$

Hence, we find another expression for R as

$$R = -4k^3\psi. \tag{2.6}$$

Finally, combining equation (2.4) and (2.6) for R, we obtain the *evolution equation* that governs ψ:

$$\psi_t + u_x\psi - 2(u+2\lambda)\psi_x + 4k^3\psi = 0. \tag{2.7}$$

Recall that $k = \sqrt{-\lambda}$. This evolution equation is important in terms of substitution in later steps to solve the KdV equation.

2.2.5 Solution of the Scattering Problem

We begin by defining the norming constant c

$$c_n(t) = \left(\int_{-\infty}^{\infty} \psi_n^2(x,t)\, dx\right)^{-1}.$$

Taking the time derivative of the inverse of the norming constant, we find that it evolves as

$$\begin{aligned}
\frac{d}{dt}(c_n^{-1}) &= \int_{-\infty}^{\infty} 2\psi\psi_t\, dx \\
&= 2\int_{-\infty}^{\infty} 2(u+2\lambda)\psi\psi_x - u_x\psi^2 - 4k^3\psi^2\, dx \quad \text{by the evolution equation (2.7)} \\
&= 2\int_{-\infty}^{\infty} 4(\psi_{xx}+\lambda\psi)\psi_x + 4\lambda\psi\psi_x - (u\psi^2)_x\, dx - 8k^3c_n^{-1} \quad \text{by substituting equation (2.2)} \\
&= -8k^3c_n^{-1}.
\end{aligned}$$

Solving this ordinary differential equation, we determine that the norming constant behaves as

$$c_n(t) = c_n(0)e^{8k_n^3 t}.$$

Consider the expression $\frac{R}{\psi}$ evaluated as x approaches infinity:

$$\frac{R}{\psi} = \lim_{x\to\infty}\left(\frac{R}{\psi}\right) = \frac{(b_t - 4ik\lambda b)e^{ikx} + 4ik\lambda e^{-ikx}}{be^{ikx} + e^{-ikx}}.$$

Observing that $\frac{R}{\psi}$ is solely a function of time, we deduce its value:

$$\frac{R}{\psi} = 4ik^3.$$

Also:

$$b_t = 8ik^3 b.$$

By solving the above ordinary differential equation, we find the scattering data, namely the reflection coefficient b defined in equation (2.3) as

$$b(k,t) = b(k,0)e^{8ik^3 t}.$$

Similarly, when $\frac{R}{\psi}$ is evaluated as x approaches negative infinity, we find:

$$4ik^3 = \frac{R}{\psi} = \lim_{x\to-\infty}\left(\frac{R}{\psi}\right) = \frac{a_t}{a} + 4ik^3.$$

This condition implies that the transmission coefficient a evolves in time as

$$a_t = 0, \quad a(k,t) = a(k,0).$$

2.2.6 The Inverse Scattering Problem

The inverse scattering problem aims to find $u(x,t)$ from scattering data (k_n, c_n, b) for $t > 0$. The expression for $u(x,t)$ is given by (see [2])

$$u(x,t) = -2\frac{\partial}{\partial x}K(x,x,t),$$

where K is the unique solution to the *linear* integral equation known as the Gelfand-Levitan-Marchenko equation:

$$K(x,y,t) + B(x+y,t) + \int_x^\infty K(x,z,t)B(y+z,t)\,dz = 0 \quad \text{for} \quad y > x. \tag{2.8}$$

Here, B is defined by:

$$B(x+y,t) = \sum_{n=1}^{p} c_n(t)e^{-k_n(x+y)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k,t)e^{ik(x+y)}\,dk.$$

Notably, when $b(k,t) = 0$, the Gelfand-Levitan-Marchenko equation is reduced to solving a $p\times p$ system of linear equations! We can find the linear system as follows:
As $b(k,t) = 0$,

$$B(x+y,t) = \sum_{n=1}^{p} c_n(t)e^{-k_n(x+y)},$$

$$K(x,y,t) = \sum_{n=1}^{p} e^{-k_n y}w_n(x,t).$$

Then the Gelfand-Levitan-Marchenko equation (2.8) becomes

$$\begin{aligned}
0 &= \sum_{n=1}^{p}(w_n(x,t) + c_n(t)e^{-k_n x})e^{-k_n y} + \int_x^\infty \left(\sum_{n=1}^{p} e^{-k_n z}w_n(x,t)\right)\left(\sum_{m=1}^{p} c_m(t)e^{-k_m(y+z)}\right)dz \\
&= \sum_{n=1}^{p}(w_n(x,t) + c_n(t)e^{-k_n x})e^{-k_n y} + \sum_{n,m=1}^{p} w_n(x,t)c_m(t)e^{-k_m y}\int_x^\infty e^{-(k_n+k_m)z}\,dz.
\end{aligned}$$

Since for each n, the coefficient of $e^{-k_n y}$ must be zero, which gives the following linear system:

$$\begin{bmatrix}
1+\frac{c_1(t)e^{-2k_1x}}{2k_1} & \frac{c_1(t)e^{-(k_2+k_1)x}}{k_2+k_1} & \cdots & \frac{c_1(t)e^{-(k_p+k_1)x}}{k_p+k_1} \\
\frac{c_2(t)e^{-(k_1+k_2)x}}{k_1+k_2} & 1+\frac{c_2(t)e^{-2k_2x}}{2k_2} & \cdots & \frac{c_2(t)e^{-(k_p+k_2)x}}{k_p+k_2} \\
\vdots & \vdots & \ddots & \vdots \\
\cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
\frac{c_p(t)e^{-(k_1+k_p)x}}{k_1+k_p} & \frac{c_p(t)e^{-(k_2+k_p)x}}{k_2+k_p} & \cdots & 1+\frac{c_p(t)e^{-2k_px}}{2k_p}
\end{bmatrix}
\begin{bmatrix} w_1(x,t) \\ w_2(x,t) \\ \vdots \\ \vdots \\ \vdots \\ w_p(x,t) \end{bmatrix}
+
\begin{bmatrix} c_1(t)e^{-k_1x} \\ c_2(t)e^{-k_2x} \\ \vdots \\ \vdots \\ \vdots \\ c_p(t)e^{-k_px} \end{bmatrix}
= \mathbf{0}.$$

2.3 Examples

2.3.1 One-soliton Example: $u(x,0) = -2\,\mathrm{sech}^2\, x$

1. Scattering Equation at $t = 0$:
$$\psi_{xx} + [\lambda - (-2\,\mathrm{sech}^2\, x)]\psi = 0.$$

2. Substitute $T = \tanh x$:
$$\frac{\partial}{\partial T}\left((1-T^2)\frac{\partial \psi}{\partial T}\right) + \left(2 + \frac{\lambda}{1-T^2}\right)\psi = 0.$$
Solving this equation, we find ψ to be:
$$\psi(T) = \frac{1}{2}(1-T^2)^{\frac{1}{2}} = \frac{1}{2}\,\mathrm{sech}\, x.$$

3. Bound State: In the case of a bound state, we have:
$$\lambda_1 = -1, \quad c_1(0) = 2, \quad c_1(t) = 2e^{8t}.$$

4. Unbound States: For unbound states, we deduce the behavior of ψ:
$$\psi(x) \sim \left(\frac{ik+1}{ik-1}\right) a e^{-ikx} \text{ as } x \to \infty.$$
which gives us the remaining scattering data:
$$a(k) = \frac{ik-1}{ik+1}, \quad b(k) = 0.$$
We can then build our solution by inverse scattering:
$$\begin{aligned} B(x+y,t) &= 2e^{8t-x-y}, \\ K(x,y,t) &= w(x,t)e^{-y}, \\ w(x,t) &= -e^{4t}\,\mathrm{sech}\,(x-4t), \\ K(x,x,t) &= -e^{-x+4t}\,\mathrm{sech}\,(x-4t), \end{aligned}$$
which gives the one-soliton solution
$$u(x,t) = -2\,\mathrm{sech}^2\,(x-4t). \tag{2.9}$$
This is what we previously obtained in equation (1.2) with $c = 4$. Figure 2 presents a plot for equation (2.9).

2.3.2 Two-soliton Example: $u(x,0) = -6\,\mathrm{sech}^2\, x$

1. $p = 2$: Consider the case when there are two eigenvalues,
$$\begin{aligned} \psi_1 = \frac{1}{4}\,\mathrm{sech}^2\, x, &\quad c_1(0) = 12 \\ \psi_2 = \frac{1}{2}\tanh x\,\mathrm{sech}\, x, &\quad c_2(0) = 6 \\ &\quad b(k,0) = 0 \end{aligned}$$

2. Solution:
$$u(x,t) = -12\frac{3 + 4\cosh(2x-8t) + \cosh(4x-64t)}{(3\cosh(x-28t) + \cosh(3x-36t))^2}. \tag{2.10}$$
Figure 3 presents a plot for equation (2.10), which is a two-soliton solution.

In general, we have n-soliton solutions. For example, Figure 4 presents a plot for a seven-soliton solution with initial condition $u(x,0) = -\mathrm{sech}^2(x)$.

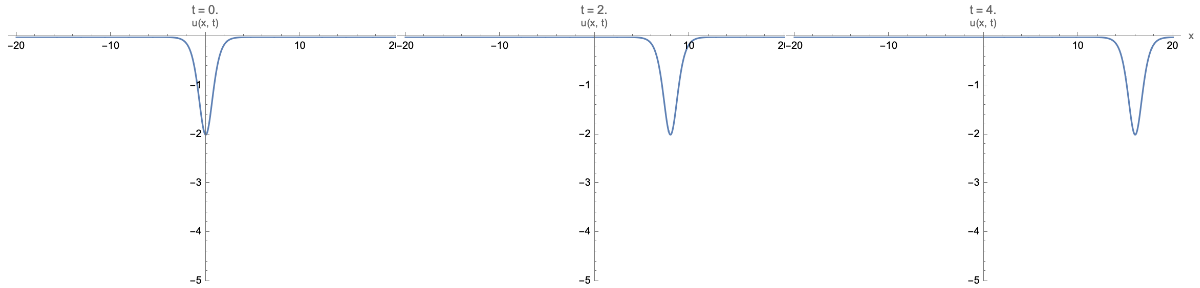

Figure 2: Time evolution of the one-soliton solution

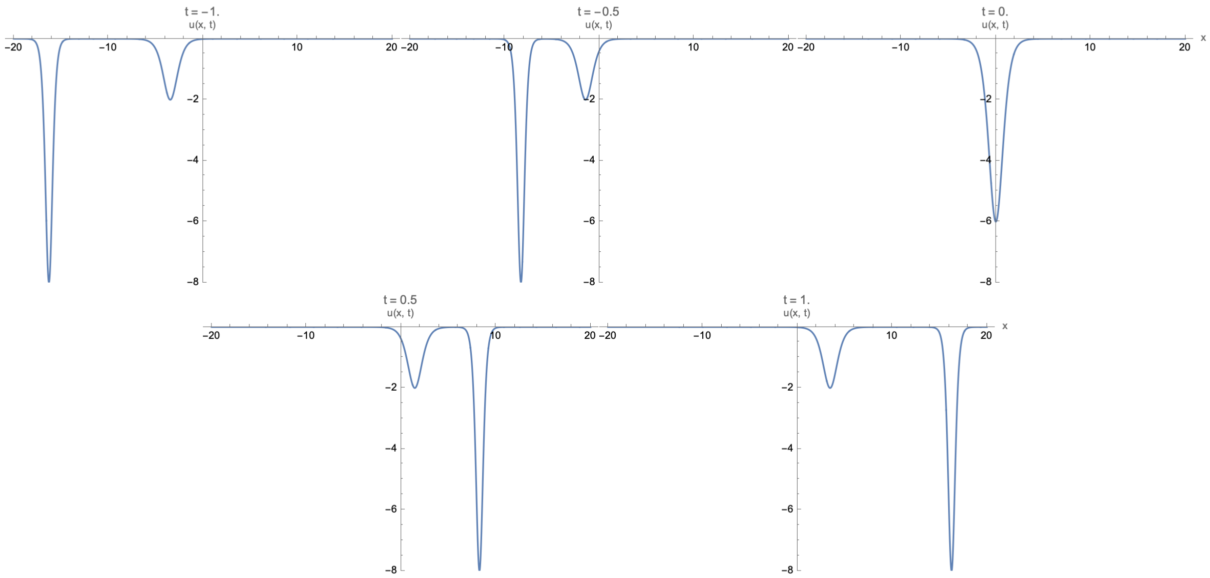

Figure 3: Time evolution of the two-soliton solution

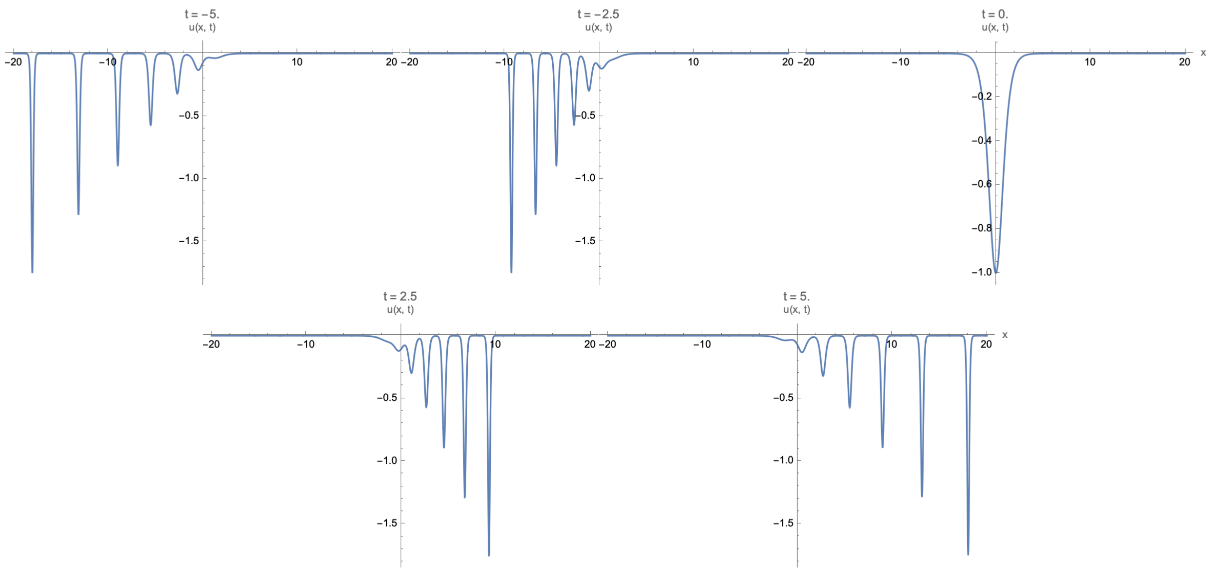

Figure 4: Time evolution of the seven-soliton solution

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References

[1] de Jager, E.M. “On the origin of the Korteweg–De Vries equation”. arXiv:math/0602661v1, 2006.

[2] I. Kay, and H.E. Moses, The determination of the scattering potential from the spectral measure function. III. Calculation of the scattering potential from the scattering operator for the one-dimensional Schrödinger equation. *Nuovo Cimento* (10), 276–304, 1956.

[3] P.G. Drazin, Solitons. Cambridge University Press, 1983.

[4] Robert M. Miura, Clifford S. Gardner, and Martin D. Kruskal, “Korteweg-de Vries Equation and Generalizations. II. Existence of Conservation Laws and Constants of Motion,” *Journal of Mathematical Physics*, vol. 9, no. 8, pp. 1204–1209, Aug. 1968.