§ Basic Riemannian Geometry

A manifold is non-collapsed at a point x：

M is a complete n-manifold。For any r such that the norm of the Riemannian curvature tensor，$|R_m| \le r^{-2}$ at all points of the metric ball B(x,r)，we have $VolB(x,r) \ge \kappa r^n$ 。

Then we say that the manifold is $\kappa - noncollapsed$ at x。

If $|R_m| \le r^{-2}$ on B(x,r) and if B(x,r) is $\kappa - noncollapsed$ ，then the injectivity radius of M at x is greater than or equal to a positive constant that only on r and κ

Instead，there is a simple equation for the evolution of volume under Ricci flow。

這裡提到兩個 non-compact canonical neighborhood

1. $\varepsilon - necks$
2. $\varepsilon - caps$

(1) Bochner formulas

$$\Delta\omega = \nabla^*\nabla\omega + \mathcal{B}^{[p]}\omega,$$

where $\mathcal{B}^{[p]}$, usually called the *Bochner operator*, is the symmetric endomorphism of the bundle of p-forms $\Lambda^p(M)$ given by $\mathcal{B}^{[p]} = \sum_{i,j=1}^n e_j \wedge (e_i \lrcorner R^M(e_j, e_i))$. Here R^M is the curvature operator on M defined by convention $R^M(X,Y) = \nabla^M_{[X,Y]} - [\nabla^M_X, \nabla^M_Y]$ and $\{e_i\}_{i=1,\cdots,n}$ denotes a local orthonormal frame of TM. In all the paper, we identify vector fields with their corresponding 1-forms through the usual musical isomorphisms.

(2) maximum principle

(3) pointwise monotonicity formulas

(4) comparison geometry

(5) integral and pointwise monotonicity formulas

§ Riemannian Geometry

Q：Given a restriction on the curvature of a Riemannian manifold，what topological conditions follow？

Myers theorem：

Cartan-Hadamard theorem：

A simply connected，complete Riemannian manifold with nonpositive sectional curvature is diffeomorphic to R^n and each exponential map is a diffeomorphism。

R. Hamilton 1982：

If (M ,g) is a closed 3-manifold with positive Ricci curvature，then it is diffeomorphic to a spherical space form。

That is， M^3 admits a metric with constant positive sectional curvature。

(closed means compact without boundary)

William Thurston：Geometrization Conjecture

Every closed 3-manifold admit a geometric decomposition。

(M, g)， $g(x): T_xM \times T_xM \to R$

$g = g_{ij} dx^i \otimes dx^j$ where $g_{ij} := g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$

$\varphi : N^m \to M^n$ is a smooth immersion，then pull back g to N

$(\varphi^* g)(V, W) := g(\varphi_* V, \varphi_* W)$

Levi-Civita connection(or covariant derivative) ∇ ， $\nabla_X Y =$

1. compatible with the metric $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$
2. torsion free
3. Lie bracket $[X,Y] = \nabla_X Y - \nabla_Y X$

Christoffel symbols $\Gamma_{ij}^k =$

$\gamma : (a,b) \to M$ is a path

A vector field X is parallel along γ if $\nabla_{\dot{\gamma}} X = 0$

A path γ is a geodesic if the unit tangent vector field is parallel along γ。

Riemann curvature tensor $R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

The components of the (3,1)-tensor are defined by

$$R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \frac{\partial}{\partial x^k} := R_{ijk}^l \frac{\partial}{\partial x^l}$$

$$R_{ijk}^l = \frac{\partial \Gamma_{jk}^l}{\partial x^i} - \frac{\partial \Gamma_{ik}^l}{\partial x^j} + \sum_m \Gamma_{jk}^m \Gamma_{im}^l - \sum_m \Gamma_{ik}^m \Gamma_{jm}^l$$

$$R_{ijkl} = g_{lm} R_{ijk}^{m} = < R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l} >$$

If $P \subset T_x M$ is a 2-plane，then the sectional curvature
$K(P) := < R(e_1, e_2) e_2, e_1 >$，where $\{e_1, e_2\}$ is an orthonormal basis of P

Exercise
Show that if X and Y are any two vectors spanning P，then

$$K(P) = \frac{< R(X,Y),Y,X >}{|X|^2 |Y|^2 - < X,Y >^2}$$

Ricci tensor Ric is the trace of the Riemann curvature tensor

$$R_{jk} = \sum_i R_{ijk}^{i}$$

Scalar curvature is the trace of the Ricci tensor

$R = \sum_i Ric(e_i, e_i)$ in local coordinate $R = g^{ij} R_{ij}$

Exercise
Show that the trace of a symmetric 2-tensor α is given by the following formula

$$Trace_g(\alpha) = \frac{1}{\omega_n} \int_{S^{n-1}} \alpha(V,V) d\rho(V)$$

Where S^{n-1} is the unit (n-1)-sphere，$n\omega_n$ its volume，and $d\rho$ its volume form。

Hint

There exists an orthonormal basis $\{e_i\}_{i=1}^{n}$ such that $\alpha = \sum_i \lambda_i e_i^* \otimes e_i^*$。

Furthermore，$Trace_g(\alpha) = \sum_i \lambda_i$ and $\frac{1}{\omega_n} \int_{S^{n-1}} \langle V, e_i \rangle^2 d\sigma(V) = 1$

定義一個(r, s) tensor 的 covariant derivative :=
$\nabla_X \beta$
則$\nabla g = 0$即為 g 與 metric 相容

Exercise

For a 1-form X， $\nabla_i X_j = \frac{\partial X_j}{\partial x^i} - \Gamma_{ij}^k X_k$

Lie derivative of a tensor with respect to X

$$L_X \alpha := \lim_{t\to 0} \frac{\alpha - (\varphi_t)_* \alpha}{t} = \lim_{t\to 0} \frac{\varphi_t^* \alpha - \alpha}{t} = \frac{d}{dt}\Big|_{t=0} \varphi_t^* \alpha$$

1. if f is a function then $L_X f = Xf$
2. if Y is a victor field then $L_X Y = [X, Y]$
3. if α, β are tensors then $L_X(\alpha \otimes \beta) = (L_X \alpha) \otimes \beta + \alpha \otimes L_X \beta$

a diffeomorphism $\psi : (M^n, g) \to (N^n, h)$ is an isometry if $\psi^* h = g$

Killing vector field X on (M, g) if $L_X g = 0$

A vector field X is complete if there is 1-parameter group of diffeomorphisms $\{\varphi_t\}_{t\in R}$ generated by X

If M is closed，then any smooth vector field is complete。

Exercise

Lie derivative of the metric

Show that $(L_X g)(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X)$ and in local coordinates this implies

$$(L_X g)_{ij} = \nabla_i X_j + \nabla_j X_i$$

Riemann curvature tensor

$$R_{ijkl} = -R_{ijlk} = -R_{jikl} = R_{klij}$$

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0$$

$$\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0$$

$$2g^{ij}\nabla_i R_{jk} = \nabla_k R \Leftrightarrow div(\mathrm{R}c - \frac{1}{2}Rg) = 0$$

Exercise

Show that for any diffeomorphism $\varphi : M \to M$, tensor α , and vector field X

$$\varphi^*(L_X \alpha) = L_{\varphi^*}(\varphi^* \alpha)$$

And if $f : M \to R$, then $\varphi^*(grad_g f) = grad_{\varphi^* g}(f \circ \varphi)$

SOLUTION. Let $\psi(t)$ denote the 1-parameter group of diffeomorphisms generated by X:

$$\begin{aligned}\varphi^*(\mathcal{L}_X \alpha) &= \varphi^*\left(\lim_{t\to 0} \frac{\psi(t)^* \alpha - \alpha}{t}\right) \\ &= \lim_{t\to 0} \frac{\left(\varphi^{-1} \circ \psi(t) \circ \varphi\right)^* \varphi^* \alpha - \varphi^* \alpha}{t} = \mathcal{L}_Y(\varphi^* \alpha)\end{aligned}$$

where Y is the vector field generating the 1-parameter group of diffeomorphisms $\varphi^{-1} \circ \psi(t) \circ \varphi$. Now

$$\begin{aligned}Y(x) &= \left.\frac{d}{dt}\right|_{t=0} \varphi^{-1} \circ \psi(t) \circ \varphi(x) = \left(\varphi^{-1}\right)_* \left.\frac{d}{dt}\right|_{t=0} \psi(t) \circ \varphi(x) \\ &= \left(\varphi^{-1}\right)_* (X(\varphi(x))) = (\varphi^* X)(x).\end{aligned}$$

For any $x \in M^n$ and $X \in T_{\varphi(x)} M^n$ we have

$$\begin{aligned}\left\langle \varphi^*(\mathrm{grad}_g f), \varphi^* X \right\rangle_{\varphi^* g}(x) &= \left\langle \mathrm{grad}_g f, X \right\rangle_g (\varphi(x)) \\ &= (Xf)(\varphi(x)) = (\varphi^* X)(f \circ \varphi)(x).\end{aligned}$$

THEOREM 1.36 (Schoen-Yau). *If (M^n, g) is a simply connected, locally conformally flat, complete Riemannian manifold, then there exists a one-to-one conformal map of (M^n, g) into the standard sphere S^n.*

§ Cartan structure equations

Let $\{e_i\}$ be a local orthonormal frame field，the dual orthonormal basis $\{\omega^i\}$ with

$\omega^i(e_j)=\delta_i^j$

We may write the metric as $g=\sum_{i=1}^{n}\omega^i\otimes\omega^i$

The connection 1-form ω_i^j are $\nabla_X e_i := \sum_{j=1}^{n}\omega_i^j(X)e_j$

The curvature 2-form Ω_i^j is defined as $R(X,Y)e_i := \sum_{j=1}^{n}\Omega_i^j(X,Y)e_j$

定理

1. $d\omega^i=\sum_j \omega^j\wedge\omega_j^i$

2. $\omega_i^j+\omega_j^i=0$

3. $\Omega_i^j=d\omega_i^j-\sum_k \omega_i^k\wedge\omega_k^j$

The Laplacian

$$\Delta := div\nabla = g^{ij}\nabla_i\nabla_j = g^{ij}\left(\frac{\partial^2}{\partial x^i\partial x^j}-\Gamma_{ij}^k\frac{\partial}{\partial x^k}\right)$$

In Euclidean space $\Delta=\sum_{i=1}^{n}\frac{\partial^2}{\partial (x^i)^2}$ and the heat equation is $(\frac{\partial}{\partial t}-\Delta)u=0$

Especially since the Ricci flow is like a heat equation，we shall often encounter the Laplacian and heat operator。

We take this opportunity to define the **divergence** of a $(p,0)$-tensor as

$$\text{(1.58)}\qquad \operatorname{div}(\alpha)_{i_1\cdots i_{p-1}} \doteqdot g^{jk}\nabla_j\alpha_{ki_1\cdots i_{p-1}} = \nabla_j\alpha_{ji_1\cdots i_{p-1}}.$$

In particular if X is a 1-form, then

$$\operatorname{div}(X)=g^{ij}\nabla_i X_j.$$

§ 2.6 integration by parts

Stokes theorem $\int_M d\alpha = \int_{\partial M} \alpha$

The divergence theorem

Let (M, g) be a compact Riemannian manifold。If X is a 1-form，then

$$\int_M div(X) d\mu = \int_{\partial M} < X, v > d\sigma$$

Hear v is the unit outward normal，$d\mu$ denotes the volume form of g，and $d\sigma \doteq i_v(d\mu)$ is the volume form of the boundary ∂M with respect to the induced metric。

PROOF. Define the $(n-1)$-form α by

$$\alpha = \iota_X(d\mu).$$

Using $d^2 = 0$ we compute

$$d\alpha = d \circ \iota_X(d\mu) = (d \circ \iota_X + \iota_X \circ d)(d\mu) = \mathcal{L}_X(d\mu) = \operatorname{div}(X) d\mu,$$

where to obtain the last equality, we may compute in an orthonormal frame $e_1, \dots, e_n$:

$$\begin{aligned} \mathcal{L}_X(d\mu)(e_1, \dots, e_n) &= \sum_{i=1}^n d\mu(e_1, \dots, \nabla_{e_i} X, \dots, e_n) \\ &= \operatorname{div}(X) d\mu(e_1, \dots, e_n). \end{aligned}$$

Now Stokes' theorem implies

$$\int_M \operatorname{div}(X) d\mu = \int_M d\alpha = \int_{\partial M} \alpha = \int_{\partial M} \iota_X(d\mu) = \int_{\partial M} X(\nu) d\sigma,$$

and the theorem is proved. □

Exercise From the divergence theorem

1. On a closed manifold，$\int_M \Delta u d\mu = 0$

2. On a compact manifold $\int_{M^n} (u\Delta v - v\Delta u) d\mu = \int_{\partial M^n} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) d\sigma.$

   In particularly，on a closed manifold $\int_M u\Delta v d\mu = \int_M v\Delta u d\mu$

3. Show that if f is a function，and X is a 1-form，then

   $$\int_M f div(X) d\mu = -\int_M < \nabla f, X > d\mu + \int_{\partial M} f < X, v > d\sigma$$

§ 3 Laplacian and Hessian comparison theorems
Laplacian comparison theorem
Hessian comparison theorem
Volume comparison theorem
Rauch comparison theorem

Jacobi field J is a variation of geodesic and satisfies the Jacobi equation。

Cut point to p along γ

Definition
A point $x \in M$ is a conjugate point of $p \in M$ if x is a singular value of $\exp_p : T_pM \to M$。That is ，$x = \exp_p(V)$ for some $V \in T_pM$，where $(\exp_p)_* : T_V(T_pM) \to T_{\exp_{(p)}(V)}M$ is singular。

Equivalently ，$\gamma(r)$ is a conjugate point to p along γ if there is a nontrivial Jacobi field along γ vanishing at the endpoints。

§ 3.1 Laplacian comparison theorem p.57~60
這裡有幾個 comparison theorem 暫略

§ 3.2 Cheeger-Gromoll splitting theorem and manifolds of nonnegative curvature
In the study of manifolds with nonnegative curvature，often the manifolds split as the product of lower dimensional manifold with a line。
Recall that a geodesic line is a unit speed geodesic $\gamma : (-\infty, \infty) \to M$ such that the distance between any points on γ is the length of the arc of γ between those two points。That is ，for any $s_1, s_2 \in (-\infty, \infty)$，$d(\gamma(s_1), \gamma(s_2)) = |s_2 - s_1|$

這裡有一個 Busemann function distance function p.61

PROPOSITION 1.75 (Mean Value Inequality for $\mathrm{Rc} \geq 0$). *If (M^n, g) is a complete Riemannian manifold with $\mathrm{Rc} \geq 0$ and if f is a Lipschitz function bounded from above and with $\Delta f \geq 0$ in the sense of distributions (**subharmonic**), then for any $x \in M^n$ and $0 < r < \mathrm{inj}(x)$*

$$f(x) \leq \frac{1}{\omega_n r^n} \int_{B(x,r)} f d\mu$$

where ω_n is the volume of the unit euclidean n-ball.

THEOREM 1.77 (Cheeger-Gromoll 1971). *Suppose (M^n, g) is a complete Riemannian manifold with $\mathrm{Rc} \geq 0$ and suppose there is a geodesic line in M^n. Then (M^n, g) is isometric to $\mathbb{R} \times (N^{n-1}, h)$ with the product metric, where (N^{n-1}, h) is a Riemannian manifold.*

REMARK 1.78. *In the study of the Ricci flow on 3-manifolds one of the primary singularity models is the round cylinder $S^2 \times \mathbb{R}$. This singularity model corresponds to neck pinching.*

A submanifold $S \subset M^n$ is **totally convex** if for every $x, y \in S$ and any geodesic γ (not necessarily minimal) joining x and y we have $\gamma \subset S$. We say that S is **totally geodesic** if its second fundamental form is zero. In particular, a path in S is a geodesic in S if and only if it is a geodesic in M^n.

Given a noncompact manifold (M^n, g) we say that a submanifold is a **soul** if it is a closed, totally convex, totally geodesic submanifold such that M^n is diffeomorphic to its normal bundle.

Generalizing earlier work of Gromoll-Meyer 1969 [**240**], Cheeger-Gromoll 1972 [**105**] (see also Poor 1974 [**429**]) proved the following.

THEOREM 1.79 (Soul). *Let (M^n, g) be a complete Riemannian manifold with nonnegative sectional curvature. Then there exists a soul. If the sectional curvature is positive, then the soul is a point (e.g., M^n is diffeomorphic to $\mathbb{R}^n$.)*

Furthermore, Sharafutdinov [**459**] proved that any two souls are isometric. In 1994 Perelman [**414**] proved the following result.

THEOREM 1.80 (Soul Conjecture). *If (M^n, g) is a complete Riemannian manifold with nonnegative sectional curvature everywhere and positive sectional curvature at some point, then the soul is a point.*

3.3. Hessian comparison theorem. The following roughly says that the larger the curvature, the smaller the Hessian of the distance function.

PROPOSITION 1.82 (Hessian comparison theorem). *Let $i = 1, 2$. Let (M_i^n, g_i) be complete Riemannian n-manifolds, let $\gamma_i : [0, L] \to M_i^n$ be geodesics parametrized by arc length such that γ_i does not intersect the cut locus of $\gamma_i(0)$, and let $d_i \doteqdot d(\cdot, \gamma_i(0))$. If for all $t \in [0, L]$ we have*

$$K_{g_1}(V_1 \wedge \dot{\gamma}_1(t)) \geq K_{g_2}(V_2 \wedge \dot{\gamma}_2(t))$$

for all unit vectors $V_i \in T_{\gamma_i(t)} M_i^n$ perpendicular to $\dot{\gamma}_i(t)$, then

$$\nabla\nabla d_1(X_1, X_1) \leq \nabla\nabla d_2(X_2, X_2)$$

for all $X_i \in T_{\gamma_i(t)} M_i^n$ perpendicular to $\dot{\gamma}_i(t)$ and $t \in (0, L]$.

§ 4 Geodesic polar coordinates

§ 4.1 Exponential map and geodesic coordinates expansion of the metric and volume form

§ 4.2 Geodesic polar coordinates and Jacobian of the exponential map

Gauss lemma

§ 4.3 The second fundamental form of the distance spheres and the Ricatti equation

Exercise 1.90

Evolution of mean curvature for a hypersurface flow

§ 4.4 Comparison with space forms and Bishop volume comparison theorem

Exercise 1.92

Curvature of a rotationally symmetric metric

Lemma 1.93

Mean curvature of distance spheres comparison

§ 4.5 Mean value inequality，Laplacian and Hessian comparison theorem

Theorem

Rauch comparison theorem p.76

Exercise 1.100. *Show that the Rauch Comparison Theorem may be used to prove the Hessian Comparison Theorem. Similarly, show that the Bishop Volume Comparison Theorem implies the Laplacian Comparison Theorem.*

§ 5 First and second variation of arc length and energy formulas p.76

With the idea of minimizing in a homotopy class of an element of $\pi_1(M^n)$ and some simple linear algebra, one also obtains:

Theorem 1.109 (Synge). *If (M^n, g) is an even-dimensional, orientable, closed Riemannian manifold with positive sectional curvature, then M^n is simply connected.*

§ 5.2 Energy

The energy functional of a path： $E(\gamma) := \int_a^b \left| \frac{d\gamma}{ds}(s) \right|^2 ds$

The critical points of the energy functional，along all paths fixing two endpoints，are the constant speed geodesics γ satisfy $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$

§ 5.3 Jacobi fields

Geodesic variations should minimize the second variation among all variations with given endpoint values。

§ 6 Geometric application of second variation

§ 6.1 Toponogov comparison theorem

§ 6.2 Long geodesics

§ 7 *Green's* function

§ 8 Comparison theory for the heat kernel(or heat equation)

§ 9 Parametrix for the heat equation

§ 10 Eigenvalues and eigenfunctions of the Laplacian

Let (M^n, g) be a complete Riemannian manifold. Let $L^2(M^n, g)$ denote the Hilbert space of measurable square integrable functions with the L^2-**inner product**:

$$\langle f, h\rangle_{L^2} = \int_{M^n} f h d\mu.$$

§ 11 The determinant of the Laplacian

§ 12 Monotonicity for harmonic functions and maps

§ 13 Lie groups and left invariant metric

§ 14 Bieberbach theorem