§ Introduction

Q：Given a restriction on the curvature of a Riemannian manifold，what topological conditions follow？

黎曼流形的曲率給個限制，它的拓撲條件會如何？

Myers theorem：Jacobi 場與二階變分式的應用。(大域微分幾何)

古典曲面論中的 Bonnet 定理：一個完全(complete)曲面 M，若其高斯曲率 $K \geq K_0 > 0$，則 M 必為緊緻(compact)且其直徑 diamM $\leq \dfrac{\pi}{\sqrt{K_0}}$

例如 球面 $S^2(R)$，R 為半徑，則 $K = \dfrac{1}{R^2}$，那麼它的 $diamS^2(R) = \pi R = \dfrac{\pi}{\sqrt{K}}$ 。

在抽象曲面上，所謂直徑指的是最遠兩點的曲面距離。

Bonnet-Myers 定理把這個古典的正曲率曲面定理推廣到高維的黎曼流形。

Theorem 1.1 (Bonnet-Myers). *Let (M, g) be a complete Riemannian manifold whose Ricci curvature satisfies*

$$\mathrm{Ric}(X_p) \geq (m-1)\kappa$$

for all $X_p \in SM$, where κ is a positive constant independent of X_p. Then M is compact, and its diameter is bounded by

$$\mathrm{diam}(M) := \sup_{p,q \in M} \mathrm{dist}(p,q) \leq \frac{\pi}{\sqrt{\kappa}}.$$

$\kappa = K_0 g$，g 是 M 的 Riemannian metric $\Rightarrow diamM \leq \dfrac{\pi}{\sqrt{K_0}}$

Topological consequences are that the manifold is compact and has finite fundamental group。

Cartan-Hadamard theorem：

A simply connected，complete Riemannian manifold with nonpositive sectional curvature is diffeomorphic to R^n and each exponential map is a diffeomorphism。

設黎曼流形 M 為 complete，且 $K_M \leq 0$，其中 K_M 表示 M 的任意截曲率，則

(1) 局部行為 M 中任兩點都不互相 conjugate，且因此 $Exp_p : T_pM \to M$ 為 local diffeomorphism。

(2) 大域行為 若 M 又為單連通 則 $M \overset{diffeo}{\approx} R^n$

In the subject of Ricci flow one starts with a Riemannian manifold and deforms the metric in the direction of minus the Ricci tensor。
The underlying differentiable manifold stays the same。
Here one hopes to prove that the geometry of the metric improves as it evolves。

§
R. Hamilton 1982：
If (M ,g) is a closed 3-manifold with positive Ricci curvature，then it is diffeomorphic to a spherical space form。
That is，M admits a metric with constant positive sectional curvature。
(closed means compact without boundary)

And use Ricci flow to prove the Geometrization Conjecture (William Thurston)：
Every closed 3-manifold admit a geometric decomposition。

A corollary of the Geometrization conjecture is the Poincare conjecture。