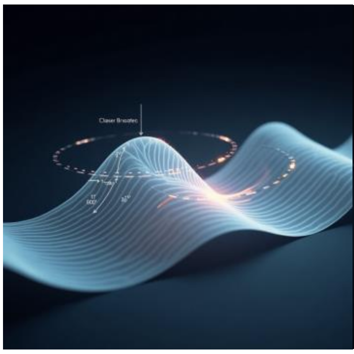
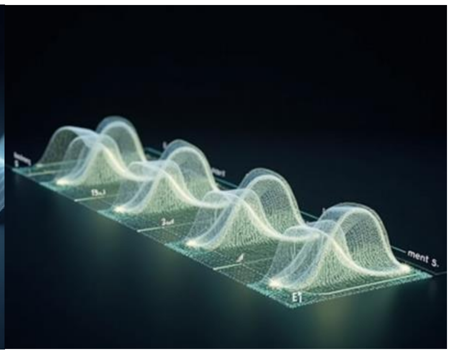
§ 推導 sine-Gordon 方程

標架微分方程

$$\begin{cases} d\vec{e_1} = \vec{e_2}d\sigma + (f\cos\psi\vec{e_2} + f\sin\psi\vec{e_3})du \\ d\vec{e_2} = (-\vec{e_1} + k\vec{e_3})d\sigma + (-f\cos\psi\vec{e_1} + v\vec{e_3})du \\ d\vec{e_3} = -k\vec{e_2}d\sigma + (-f\sin\psi\vec{e_1} - v\vec{e_2})du \end{cases}$$

$$k = -\psi_\sigma + \frac{f_\sigma}{f}\cot\psi$$

到 $v = f_\sigma \csc\psi$ $\Rightarrow$ sine-Gordon 方程。

$$0 = -k_u + v_\sigma + f\sin\psi$$

$$d(d\vec{e_1}) = d\vec{e_2} \wedge d\sigma + d(f\cos\psi\vec{e_2} + f\sin\psi\vec{e_3}) \wedge du$$

其中 $d\vec{e_2} \wedge d\sigma = [(-f\cos\psi\vec{e_1} + v\vec{e_3})du] \wedge d\sigma$ ，let $V = f\cos\psi\vec{e_2} + \mathrm{f}\sin\psi\vec{e_3}$

$$d(f\cos\psi) = (f_\sigma\cos\psi - f\sin\psi\psi_\sigma)d\sigma + (f_u\cos\psi - f\sin\psi\psi_u)du$$

$$d(f\sin\psi) = (f_\sigma\sin\psi + f\cos\psi\psi_\sigma)d\sigma + (f_u\sin\psi + f\cos\psi\psi_u)du$$

$dV = d(f\cos\psi)\vec{e_2} + f\cos\psi d\vec{e_2} + d(f\sin\psi)\vec{e_3} + f\sin\psi d\vec{e_3}$ 其 $d\sigma$ 的 component 為

$$[dV]_{d\sigma} = -f\cos\psi\vec{e_1} + (f_\sigma\cos\psi - f\sin\psi\psi_\sigma - fk\sin\psi)\vec{e_2} + (f_\sigma\sin\psi + f\cos\psi\psi_\sigma + fk\cos\psi)\vec{e_3}$$

$dV \wedge du = [dV]_{d\sigma} \wedge du = \ldots$(上面的式子)$d\sigma \wedge du$

因為 $dd(\vec{e_1}) = 0$

$\vec{e_2}$ 的係數 $f_\sigma\cos\psi - f\sin\psi\psi_\sigma - fk\sin\psi = 0$ 得到

$$k = -\psi_\sigma + \frac{f_\sigma}{f}\cot\psi$$

$\vec{e_3}$ 的係數 $-v + f_\sigma\sin\psi + f\cos\psi\psi_\sigma + fk\cos\psi = 0$ ，將 k 代入化簡得

$v = f_\sigma \csc\psi$

由 $d(d\overrightarrow{e_3}) = 0$ 同理，經過一繁複之計算推出

$0 = -k_u + v_\sigma + f\sin\psi$

$[\psi_\sigma - \frac{f_\sigma}{f}\cot\psi]_u + [f_\sigma \csc\psi]_\sigma + f\sin\psi = 0$

假設 f 是一常數，則 $\psi_{\sigma u} + f\sin\psi = 0$ 是 sine-Gordon 方程。