§ $div_M N = 2H$

考慮 Weingarten map $W(X) = \nabla_X N(p)$

$$div_M N := <\nabla_{e_1} N, e_1> + <\nabla_{e_2} N, e_2> = <W(e_1), e_1> + <W(e_2), e_2> = tr([W])$$

§ Exercises

$$H = \frac{Ge + Eg - 2Ff}{2(EG - F^2)}$$

$z = \sinh x \sqrt{1 - (\frac{y}{\cosh x})^2}$ ，求 mean curvature H=

$\sinh x = \frac{e^x - e^{-x}}{2}, \cosh x = \frac{e^x + e^{-x}}{2}$ ， $\cosh^2 x - \sinh^2 x = 1$

$$X(x, \theta) = (x, \cosh x \sin\theta, \sinh x \cos\theta)$$
$$X_x = (1, \sinh x \sin\theta, \cosh x \cos\theta)$$
$$X_\theta = (0, \cosh x \cos\theta, -\sinh x \sin\theta)$$

$$E = X_x \cdot X_x = 1 + \sinh^2 x \sin^2\theta + \cosh^2 x \cos^2\theta$$

$$F = X_x \cdot X_\theta = 0$$

$$G = X_\theta \cdot X_\theta = \cosh^2 x \cos^2\theta + \sinh^2 x \sin^2\theta$$

$$X_x \times X_\theta = (-\sinh^2 x - \cos^2\theta, \sinh x \sin\theta, \cosh x \cos\theta)$$

$$|X_x \times X_\theta|^2 =$$

$$N = \frac{X_x \times X_\theta}{|X_x \times X_\theta|} = \frac{\ldots}{\sqrt{(\sinh^2 x + \cos^2\theta)(\sin^2 x + \cos^2\theta + 1)}}$$

$$X_{xx} = (0, \cosh x \sin\theta, \sinh x \cos\theta)$$
$$X_{x\theta} = (0, \sinh x \cos\theta, -\cosh x \sin\theta)$$
$$X_{\theta\theta} = (0, -\cosh x \sin\theta, -\sinh x \cos\theta)$$

$$e = \frac{\cosh x \sinh x}{\sqrt{(\sinh^2 x + \cos^2\theta)(\sinh^2 x + \cos^2\theta + 1)}}$$

$$g = \frac{-\cosh x \sinh x}{\sqrt{(\sinh^2 x + \cos^2\theta)(\sinh^2 x + \cos^2\theta + 1)}}$$

$$H = \frac{eG + gE}{2EG} = \frac{e(\text{G-E})}{2EG} = \frac{-e}{2\text{EG}}$$

For fixed x，$(\frac{y}{\cosh x})^2+(\frac{z}{\sinh x})^2=\cos^2\theta+\sin^2\theta=1$

This symmetry implies the surface is rotationally invariant around the *x*-axis，a hallmark(標誌) of minimal surfaces like the catenoid。

The term e contains coshxsinhx in the numerator，but for a minimal surface，these terms must inherently(本質上) cancel globally due to the surface's geometric constraints (e.g., the identity $\cosh^2 x-\sinh^2 x=1$)

Catenoid：

$$\begin{cases} x=c\cosh(\frac{v}{c})\cos u \\ y=c\sinh(\frac{v}{c})\sin u \\ z=v \end{cases}$$

For fixed x，$(\frac{y}{\cosh x})^2+(\frac{z}{\sinh x})^2=\cos^2\theta+\sin^2\theta=1$ is a catenoid。

§ Prove that $div_M N=2H=\sum_{i=1}^{2}\left\langle\nabla_{e_i}N,e_i\right\rangle$，where e_i is an orthonormal frame for M。

M 是嵌入 R^3 中的光滑曲面，N 是 M 上的單位法向量。

$div_M(N)$表示沿著曲面 M 的散度計算。

$\{e_1,e_2\}$是T_pM 的一組標準正交基，N 在每一點都垂直T_pM

$$div_M(N):=<\nabla_{e_1}N,e_1>+<\nabla_{e_2}N,e_2>$$

對任意切向量$W\in T_pM$ ，方向導數$\nabla_W N\perp N(\because\|N\|=1)$ ，因此$\nabla_W N\in T_pM$

Shape operator $S:T_pM\to T_pM, S(W)=\nabla_W N$ 則

$$div_M N=<S(e_1),e_1>+<S(e_2),e_2>=tr(S)$$

$$H=\frac{1}{2}tr(S),\therefore div_M N=2H$$