§ mean curvature

$\kappa = \frac{dT}{ds} = \kappa_n N + \kappa_g U$

$e_1, e_2$ 是對應主曲率的主方向，$e_1 \perp e_2$

$I = \mathrm{E}du^2 + 2Fdudv + Gdv^2$ $[I] = \begin{pmatrix} E & \mathrm{F} \\ \mathrm{F} & \mathrm{G} \end{pmatrix}$, $[II] = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$

$II = Ldu^2 + 2Mdudv + Ndv^2$

形狀算子 $S = [I]^{-1}[II]$ (Weingarten map)的兩個 eigenvalues 即 $\kappa_1, \kappa_2$

則 $H = \frac{1}{2} tr(S), K = \det(S)$

§ Euler theorem

$v = \cos\theta e_1 + \sin\theta e_2$ 則 Euler theorem：$\kappa_n = \cos^2\theta\kappa_1 + \sin^2\theta\kappa_2$

$II(e_1, e_1) = \kappa_1, II(e_2, e_2) = \kappa_2, II(e_1, e_2) = 0$

$\kappa_n = II(v, v) = II(\cos\theta e_1 + \sin\theta e_2, \cos\theta e_1 + \sin\theta e_2)$

$$= \cos^2\theta \mathrm{II}(e_1, e_1) + 2\cos\theta\sin\theta II(e_1, e_2) + \sin^2\theta II(e_2, e_2)$$
$$= \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$$

§ $e_1 \perp e_2$

是根據線性代數中的譜定理，一個實對稱矩陣（或算子）的特徵向量，若對應不同的特徵值，則它們必然彼此正交。

A 是實對稱矩陣，$A^T = A \Rightarrow \langle Au, v \rangle = \langle u, Av \rangle$ for any u，v

( 因為 $\langle x, y \rangle = x^T y$，$\langle Au, v \rangle = (Au)^T v = u^T A^T v = u^T Av = \langle u, Av \rangle$ )

設 $Ax = \lambda x, Ay = \mu y$，$\langle x, Ay \rangle = \langle x, \mu y \rangle = \mu \langle x, y \rangle$，$\langle Ax, y \rangle = \langle \lambda x, y \rangle = \lambda \langle x, y \rangle$

$(\mu - \lambda)\langle x, y \rangle = 0$ 所以 $x \perp y$

S 算子是自共軛的，即 $\langle S(u), v \rangle = \langle u, S(v) \rangle$

$S(e_1) = \kappa_1 e_1, S(e_2) = \kappa_2 e_2$

所以$e_1 \perp e_2$

§

$$\kappa_n = \frac{\mathrm{II}(v,v)}{I(v,v)} = \frac{v^T[II]v}{v^T[I]v}$$

利用 Lagrange Multipliers 在約束條件$v^Y[I]v=1$ (即 v 為單位向量)下，求

$f(v)=v^{\mathrm{T}}[\mathrm{II}]v$ 的極值

設$L(v,\lambda)=v^T[II]v-\lambda(v^T[I]v-1)$ 對 v 為分後取其值為 0

$$\frac{\partial L}{\partial v}=2[II]v-2\lambda[\mathrm{I}]=0$$

$$[II]v=\lambda[I]v$$

$[I]^{-1}[II]v=\lambda v$ 即$Sv=\lambda v$

因為法曲率的極值發生在這個方向，所以極值$\lambda$ 依定義就是 $\kappa_1,\kappa_2$

§ $S^2$

$$r(\theta,\phi)=(R\sin\theta\cos\phi, R\sin\theta\sin\phi, R\cos\theta)$$

$$[I]=\begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix},[II]=\begin{pmatrix} R & 0 \\ 0 & R\sin^2\theta \end{pmatrix}$$

$S=[I]^{-1}[II]=\begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{R} \end{pmatrix}$ 所以$H=\frac{1}{R}, K=\frac{1}{R^2}$

§

由 Euler 定理$H=\frac{1}{2\pi}\int_0^{2\pi}\kappa_n(\theta)d\theta=\frac{1}{2\pi}\int_0^{2\pi}(\cos^2\theta\kappa_1+\sin^2\theta\kappa_2)d\theta=...=\frac{1}{2}(\kappa_1+\kappa_2)$

§ Lagrange Multipliers(Lagrange 乘數法)

例 $x^2+4y^2=8$的約束下，求 f(x,y)=x+2y 的極值

$$L(x,y,\lambda)=(x+2y)+\lambda(x^2+4y^2-8)$$

$$\frac{\partial L}{\partial x}=1+\lambda(2x)=0 \Rightarrow x=-\frac{1}{2\lambda}$$

$$\frac{\partial L}{\partial y}=2+\lambda(8y)=0 \Rightarrow y=-\frac{1}{4\lambda}$$

代入 $x^2+4y^2=8 \quad \lambda=\pm\frac{1}{4}$

$\lambda=-\frac{1}{4}$ 時，x=2,y=1 f(2,1)=4 是極大值。

例 在 $g(x,y,z)=x^2+y^2+z^2=1, h(x,y,z)=x+2y+3z=0$ 的約束下 求 f(x,y,z)=x+y+z 的極值

$$L(x,y,z,\lambda,\mu)=(x+y+z)+\lambda(x^2+y^2+z^2-1)+\mu(x+2y+3z)$$

$$\frac{\partial L}{\partial x}=0, \frac{\partial L}{\partial y}=0, \frac{\partial L}{\partial z}=0, x^2+y^2+z^2=1, x+2y+3z=0$$

得 $\mu=\frac{3}{7} \Rightarrow x:y:z=4:1:-2$

$P_1(\frac{4}{\sqrt{21}},\frac{1}{\sqrt{21}},\frac{-2}{\sqrt{21}})$ 得 max= $\frac{3}{\sqrt{21}}$