

§ Iwasawa decomposition theorem

$$SL(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\}, \text{ 試證 } SU(2) \text{ 與 } S^1 \times R^2 \text{ 可微同胚(diffeomorphic)}$$

(1) $SL(2)$ 是一個 3 維流形(in fact a Lie group)

(2) 取 $a=p+q, d=p-q, b=r+s, c=r-s$

$$\text{則 } (p^2 + s^2) - (q^2 + r^2) = 1$$

$$\text{取 } p = \cosh t \cos \alpha, s = \cosh t \sin \alpha, q = \sinh t \cos \beta, r = \sinh t \sin \beta$$

$$\text{其中 } \cosh t = \frac{e^t + e^{-t}}{2}, \sinh t = \frac{e^t - e^{-t}}{2}$$

$$\text{則 } a = \cosh t \cos \alpha + \sinh t \cos \beta, b = \sinh t \sin \beta + \cosh t \sin \alpha$$

$$c = \sinh t \sin \beta - \cosh t \sin \alpha, d = \cosh t \cos \alpha - \sinh t \sin \beta$$

$$SU(2) \xrightarrow{f} S^1 \times R^2$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow ?$$

定理 1.3

Let G be any connected semisimple Lie group with Lie algebra \bar{g}

Then $\bar{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ (direct vector space sum)

$$G = KAN$$

That is, the mapping $(\kappa, a, n) \rightarrow \kappa a n$ is a diffeomorphism of $K \times A \times N$ onto G

If $g \in G$ we can write the decomposition $g = \kappa(g) \exp H(g) n(g)$

$$\kappa(g) \in K, H(g) \in \mathfrak{a}, n(g) \in N$$

$$\text{Lie algebra of } SL(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + d = 0 \right\} \text{ 的}$$

$$\text{Iwasawa decomposition} = \left\{ \lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mid \lambda, a, b \in R \right\} \text{ 所以}$$

$$K = \left\{ e^{\lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} = \begin{pmatrix} \cos \lambda & \sin \lambda \\ -\sin \lambda & \cos \lambda \end{pmatrix} \right\}, A = \left\{ e^{a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} = \begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix} \right\}$$

$$N = \left\{ e^{b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\},$$

$$\text{所以 } ([\lambda], a, b) \in S^1 \times R^2 \rightarrow \begin{pmatrix} \cos \lambda & \sin \lambda \\ -\sin \lambda & \cos \lambda \end{pmatrix} \begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in SL(2)$$

$0 \leq \lambda \leq 2\pi, [0] = [2\pi]$

就是所求的可微同胚

$SL(2)$ 是 R^2 上所有保持定向面積的線性變換群，它同構於 $SU(1,1)$

一般而言 一連通李群 G 與 $K \times R^n$ 可微同胚 其中 K 是 G 的極大緊緻子群 (maximal compact subgroup)

$SO(2)$ 是 $SL(2)$ 的 maximal compact subgroup

用 Gram-Schmidt 法 可以使 $SL(2)$ 與 $SO(2) \times R^2$ 可微同胚

$O(n) = \{A \in M \mid A^t A = I\}$ 正交群(orthogonal group)

$SO(n) = \{A \in O(n) \mid \det A = 1\}$ 特殊正交群(旋轉群)

李群 E8 248 維 超旋理論的產物 1887 年提出 Jeffrey Adams 、 David Vogan