Lie algebra ismorphism to left-invariant fields

1. A Lie algebra and Lie bracket

   A vector field X on G is called left-invariant if for every $g \in G$, the pushforward of X by the left translation map $L_g : G \to G$ (defined by $L_g(h) = gh$) satisfies:

   $$(L_g)_* X = X.$$

2. This means that for any $h \in G$, $X_{gh} = (L_g)_* X_h$.

Construct the isomorphism：

We define a map $\phi : \mathfrak{g} \to \mathrm{Lie}(G)$, where $\mathrm{Lie}(G)$ is the space of left-invariant vector fields on G, as follows:

$$\phi(X)(g) = (L_g)_* X,$$

for $X \in \mathfrak{g}$ and $g \in G$. Here, $(L_g)_* X$ is the pushforward of X by L_g.

Show ϕ is a vector space isomorphism：

- **Injectivity**: If $\phi(X) = 0$, then $(L_g)_* X = 0$ for all $g \in G$. In particular, at $g = e$, we have $X = 0$. Thus, ϕ is injective.
- **Surjectivity**: Given a left-invariant vector field Y on G, define $X = Y_e \in \mathfrak{g}$. Then, by left-invariance, $Y_g = (L_g)_* X$ for all $g \in G$, so $Y = \phi(X)$. Thus, ϕ is surjective.

Show that ϕ preserves the Lie bracket：

We need to show that $\phi([X, Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{g}$.

- The Lie bracket on $\mathfrak{g}$ is defined by $[X, Y] = \mathrm{ad}_X Y$, where $\mathrm{ad}_X Y$ is the derivative of the adjoint action.
- The Lie bracket of left-invariant vector fields is given by the commutator of vector fields.

Using the definition of ϕ and the properties of left-invariant vector fields, we can show that:

$$\phi([X, Y])(g) = (L_g)_*[X, Y] = [(L_g)_* X, (L_g)_* Y] = [\phi(X)(g), \phi(Y)(g)].$$

Thus, $\phi([X, Y]) = [\phi(X), \phi(Y)]$, and ϕ preserves the Lie bracket.