§ Hyperbolic Plane

習作 7.17 (3)

一個李群 G 是一個平滑流形(smooth maniflod) 同時是一個群 使得群的運算

$G\times G\to G \qquad G\to G$

$(g,h)\mapsto gh \qquad g\mapsto g^{-1}$ 都是可微映射

我們把 $H=\{(x,y)\in R^2 \mid y>0\}$ 上的每一點與可逆仿射映射(affine map)

$h:R\to R, h(t)=yt+x$ 等同(identify)，所有這樣的映射所成的集合在結合律下為一群 因此我們在 H 上引入(induce)一個群結構

試證

(a) 這個誘導的(induced)群運算為 $(x,y)\cdot(z,w)=(yz+x,yw)$

Given two affine maps g(t)=yt+x and h(t)=wt+z，we have

$(g\circ h)(t)=$ g(h(t))=g(wt+z)=ywt+yz+x

Therefore the group operation is given by $(x,y)\bullet(z,w)=$ (yz+x,yw)

The identity element is e=(0,1)，hence

$$(z,w)=(x,y)^{-1}\Leftrightarrow(yz+x,yw)=(0,1)\Leftrightarrow(z,w)=(-\frac{x}{y},\frac{1}{y})$$

Therefore the maps $(g,h)\to g\bullet h$ and $g\to g^{-1}$ are smooth hence H is a Lie group。

(b) Show that the derivative of the left translation map $L_{(x,y)}: H\to H$ at a point $(z,w)\in H$ is represented in the above coordinates by the matrix

$$\left(dL_{(x,y)}\right)_{(z,w)}=\begin{pmatrix} y & 0\\ 0 & y\end{pmatrix}.$$

Conclude that the left-invariant vector field $X^V\in\mathfrak{X}(H)$ determined by the vector

$$V=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\in\mathfrak{h}\equiv T_{(0,1)}H \qquad (\xi,\eta\in\mathbb{R})$$

is given by

$$X^V_{(x,y)}=\xi y\frac{\partial}{\partial x}+\eta y\frac{\partial}{\partial y}.$$

$g\in G$，$G\xrightarrow{L_g}G \quad L_g h=gh$ 稱為左乘運算，是一個 diffeomophism

$$(L_{(x,y)})(z,w)=(x,y)\cdot(z,w)=(yz+x,yw)=(x',y')$$

The matrix represantation is $\begin{pmatrix} \frac{\partial x'}{\partial z} & \frac{\partial x'}{\partial w} \\ \frac{\partial y'}{\partial z} & \frac{\partial y'}{\partial w} \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}$

$$\mathrm{X}^V_{(x,y)} = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} y\xi \\ y\eta \end{pmatrix} = \xi y \frac{\partial}{\partial x} + \eta y \frac{\partial}{\partial y}$$

$T_h(G) \overset{(\mathrm{L}_g)_*}{\rightarrow} T_{gh}(G)$ is a induced map

If $X \in \chi(G)$ $(L_g)_* \mathrm{X} = \mathrm{X}$ for $\forall g \in G$ then X is called left-invariant。

G 的 Lie algebra 與 left-invariant vector field space 等價。
亦即 $\eta \cong \mathrm{X_L(H)}$

(c) Given $V, W \in \mathfrak{h}$, compute $[V, W]$.

(c) If

$$V = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \quad \text{and} \quad W = \zeta \frac{\partial}{\partial x} + \omega \frac{\partial}{\partial y}$$

then

$$\left[X^V, X^W\right] = \left[\xi y \frac{\partial}{\partial x} + \eta y \frac{\partial}{\partial y}, \zeta y \frac{\partial}{\partial x} + \omega y \frac{\partial}{\partial y}\right]$$
$$= (\eta\zeta - \omega\xi) y \frac{\partial}{\partial x}.$$

Therefore

$$[V, W] = \left[X^V, X^W\right]_{(0,1)} = (\eta\zeta - \omega\xi)\frac{\partial}{\partial x}.$$

(d) Determine the flow of the vector field X^V, and give an expression for the exponential map $\exp : \mathfrak{h} \to H$.

(e) Confirm your results by first showing that H is the subgroup of $GL(2)$ formed by the matrices

$$\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$$

with $y > 0$.

(d)

$(L_g)_*[X,Y] = [(L_g)_* X, (L_g)_* Y]$ 即 $[X,Y] = [X^V, Y^V]_e$

因為 $X^V=\xi y\frac{\partial}{\partial x}+\eta y\frac{\partial}{\partial y}$，所以 X^V 的 flow 是微分方程 $\begin{cases}\dot{x}=\xi y\\ \dot{y}=\eta y\end{cases}$ 的解，

若 $\eta\neq 0$ 則 $\begin{cases} y=y_0e^{\eta t}\\ x=x_0+\frac{\xi y_0}{\eta}(e^{\eta t}-1)\end{cases}$，此時 $\exp(V)=(\frac{\xi(e^{\eta}-1)}{\eta},e^{\eta})$

若 $\eta=0$ 則 $\begin{cases} y=y_0\\ x=x_0+y_0\xi t\end{cases}$，此時 $\exp(V)=(\xi,1)$

Exp: $\eta\to H$，$(x_0,y_0)=e=(0,1),t=1$

(e) The multiplication of two such matrices is

$$\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\begin{pmatrix} w & z\\ 0 & 1\end{pmatrix}=\begin{pmatrix} yw & yz+x\\ 0 & 1\end{pmatrix},$$

which reproduces the group operation on H. Therefore H can be identified with the corresponding subgroup of $GL(2)$. A curve $c:(-\varepsilon,\varepsilon)\to H$ with $c(0)=I$ is then given by

$(x,y)\overset{\psi}{\to}\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}$，$\psi(g,h)=\psi(g)\psi(h),\psi(0,1)=I$

$\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\begin{pmatrix} z & w\\ 0 & 1\end{pmatrix}=\begin{pmatrix} yw & yz+x\\ 0 & 1\end{pmatrix}$，所以可以說 $\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}$ 是 H 的 representation

這個群運算讓 H 成為 GL(2)的子群

一個曲線 $c:(-\varepsilon,\varepsilon)\to H$，c(0)=I

則 $c(t)=\begin{pmatrix} y(t) & x(t)\\ 0 & 1\end{pmatrix}, x(0)=0, y(0)=1$，

其在 t=0 的導數 $\dot{c}(0)=\begin{pmatrix} \dot{y}(0) & \dot{x}(0)\\ 0 & 0\end{pmatrix}=\begin{pmatrix} \eta & \xi\\ 0 & 0\end{pmatrix}$ 就是在 η 的矩陣形式

其 Lie bracket 為

$$[\begin{pmatrix} \eta & \xi\\ 0 & 0\end{pmatrix},\begin{pmatrix} \omega & \zeta\\ 0 & 0\end{pmatrix}]=\begin{pmatrix} \eta & \xi\\ 0 & 0\end{pmatrix}\begin{pmatrix} \omega & \zeta\\ 0 & 0\end{pmatrix}-\begin{pmatrix} \omega & \zeta\\ 0 & 0\end{pmatrix}\begin{pmatrix} \eta & \xi\\ 0 & 0\end{pmatrix}=\begin{pmatrix} 0 & \eta\zeta-\omega\xi\\ 0 & 0\end{pmatrix}$$

$$\exp\begin{pmatrix} \eta & \xi\\ 0 & 0\end{pmatrix}=\sum_{k=0}^{+\infty}\frac{1}{k!}\begin{pmatrix} \eta & \xi\\ 0 & 0\end{pmatrix}^k=\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix} \eta & \xi\\ 0 & 0\end{pmatrix}+\frac{1}{2}\begin{pmatrix} \eta^2 & \eta\xi\\ 0 & 0\end{pmatrix}+\cdots,$$

$$\exp\begin{pmatrix}\eta & \xi \\ 0 & 0\end{pmatrix} = \begin{pmatrix}e^{\eta} & \frac{\xi(e^{\eta}-1)}{\eta} \\ 0 & 1\end{pmatrix}$$ for $\eta \neq 0$

$$\exp\begin{pmatrix}\eta & \xi \\ 0 & 0\end{pmatrix} = \begin{pmatrix}1 & \xi \\ 0 & 1\end{pmatrix},$$ for $\eta = 0$

這裡可以看到 Lie group H 的(1)Lie bracket (2)exp 的計算都變成矩陣運算。