§ 證明 O(n)是一個李群

$M \xrightarrow{f} N, T_A M \xrightarrow{df} T_{f(A)} N$

1. $p \in M$ ，若 $(df)_p$ 是蓋射(surjective) 則稱 p 是 f 的正則點(regular point)

2. $\ker(df)_p = \{B \in T_A M \mid (df)_A B = 0\}$

3. $q \in N$ ，若 $\forall p \in f^{-1}(g)$ 皆為正則點 則稱 q 為 f 的正則值

定理(regular value theorem)

$M \xrightarrow{f} N$ ，$q \in N$ 是 f 的正則值 使得 $L = f^{-1}(q) = \{p \in M \mid f(p) = q\} \neq \emptyset$

則 L 是 M 的子流形，且 $T_p L = kef(df)_p \subset T_p M$ for all $p \in L$

以下證明 O(n)=$\{A \in M \mid A^t A = I\}$ 是一個李群

$GL(n) \xrightarrow{f} S$ (S 是對稱矩陣)， $f(A) = A^t A$

$$(df)_A B = \lim_{h \to 0} \frac{f(A+hB) - f(A)}{h} = \dots = A^t B + B^t A$$

Since $A \in O(n)$ ，$A^t = A^{-1}$ ，substituting $Y = A^t B$ ，$(df)_A Y = Y + Y^t$

For any symmetric matrix S，set $Y = \frac{1}{2} S$ solves $Y^t + Y = S$ ，showing that $(df)_A$ is surjective。

Since I is a regular value，by regular value theorem O(n) is a smooth manifold。

The dimO(n)= $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$

$O(n)$ is a subgroup of $GL(n,R)$，which is a Lie group。 The group operations (multiplication and inversion) on $O(n)$ are smooth as they are restrictions of smooth operations on $GL(n,R)$。Thus，$O(n)$ is a Lie group。

Regular value theorem

Let M and N be smooth manifolds of dimensions m and n， respectively，and let $f : M \to N$ be a smooth map。A point $y \in N$ is called a **regular value** of f if for every $x \in f^{-1}(y)$, the differential $df_x : T_x M \to T_y N$ is surjective (i.e., the Jacobian matrix of f at x has full rank。

The Regular Value Theorem states：

If $y \in N$ is a regular value of f，then the preimage $f^{-1}(y)$ is a smooth submanifold of M of dimension m-n。

This theorem is widely used to construct manifolds and study their properties。For example，it is used to show that level sets of smooth functions are manifolds。

Example

1. Consider the smooth map $f : R^3 \to R$ defined by $f(x,y,z) = x^2 + y^2 + z^2$

   The derivative $df = (2x, 2y, 2z)$

   The only critical point of f is (0,0,0)，so every $c \neq 0$ is a regular value。

   For c>0，the preimage $f^{-1}(c)$ is the sphere S^2 of radius $\sqrt{c}$，which is a 2-dimensional smooth manifold。

The proof relies on the **Implicit Function Theorem**. At each point $x \in f^{-1}(y)$, the surjectivity of df_x allows us to locally express $f^{-1}(y)$ as the graph of a smooth function, ensuring it is a smooth manifold.

Application

1. Level sets

   If $f : M \to R$ is a smooth function and c is a regular value，then $f^{-1}(c)$ is a smooth hypersurface in M。

2. Lie grups

   The theorem is used to show that certain subsets of a Lie groups are submanifolds。

3. Physics：In mechanics，level sets of conserved quantities (e.g. energy) often form manifolds。