§ Lie Groups & Lie Algebra

§1 定義

Lie group G

$G \times G \to G : (a,b) \to a \cdot b$
$G \to G : a \to a^{-1}$
群運算與微分結構相容 即左邊的 mapping 是可微分的

Lie algebra V with Lie bracket

1. 雙線性
2. [A,B]=-[B,A]
3. Jacobi identity

Lie algebra(the tangent space to G at identity)扮演重要腳色，G 的群運算在 $\bar{g} = T_e G$ 上定義一種雙線性扭對稱運算 bilinear skew-symmetric operation)。
G 的許多基本性質由相關的李代數性質決定，同樣地 李代數也是研究李群結構與其表現之鑰。

§2 例子

Lie group	Lie algebra
O(n)= $\{A \in M_{n\times n} \mid A^t A = I\}$	$o(n) = \{A \in M_{n\times n} \mid A^t + A =$ (
$SL(n) = \{A \in M_{n\times n} \mid \det A = 1\}$	$sl(n) = \{A \in M_{n\times n} \mid trA =$ (

(S.Lie 稱為 infinitesimal group，H Weyl 稱為 Lie algebra)

SL(n) detA=1

$M \xrightarrow{f} R$

$A \to \det A$

Then $(df)_A B = (\det A)(trA^{-1}B)$

$(df)_I B = trB$

sl(n)= $T_e G = \ker(df)_I \to trB = 0$

即 sl(n)= $\{A \in M \mid trA = 0\}$

O(n) $A^t A = I$

$M \xrightarrow{f} S$

$A \to A$ ...

$(df)_A B = B^t A + A^t B$

$(df)_I B = B^t + B$

o(n) $= T_e G = \ker(df)_I \to B^t + B = O$

o(n)= $\{A \in M \mid A^t + A = O\}$

§3 Group action　G 是李群， DiffM 是 M 到 M 的 diffeomorphism

$G \xrightarrow{\rho} DiffM$ 滿足

1. $\rho(1) = id$
2. $\rho(gh) = \rho(g)\rho(h)$
3. $G \times M \to M \quad (g,m) \to \rho(g).m$ is a smooth map

例如

(1) GL(n,R) acts on R^n

(2) O(n,R) acts on $S^{n-1} \subset R^n$

(3) U(n) acts on $S^{2n-1} \subset C^n$

Representation V

$g \in G$ ，assign a linear map $\rho(g): V \to V$ 使得 $\rho(g)\rho(h) = \rho(gh)$

Orbit(軌道) $\forall m \in M$ ， $O_m = \{g.m | g \in G\}$

Stabilizer(isotropy subgroup 迷向子群) $G_m = \{g \in G | g.m = m\}$

§4 在物理上的應用 Alexander Kirillov

李群與李代數如何應用在解決對稱性上

If we had just one symmetry, given by some rotation $R: S^2 \to S^2$, we could consider its action on the space of complex-valued functions $C^\infty(S^2, \mathbb{C})$. If we could diagonalize this operator, this would help us study Δ_{sph}: it is a general result of linear algebra that if A, B are two commuting operators, and A is diagonalizable, then B must preserve eigenspaces for A. Applying this to pair R, Δ_{sph}, we get that Δ_{sph} preserves eigenspaces for R, so we can diagonalize Δ_{sph} independently in each of the eigenspaces.

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere. Define the Laplace operator $\Delta_{\text{sph}} : C^\infty(S^2) \to C^\infty(S^2)$ by $\Delta_{\text{sph}} f = (\Delta \tilde{f})|_{S^2}$, where $\tilde{f}$ is the result of extending f to $\mathbb{R}^3 - \{0\}$ (constant along each ray), and Δ is the usual Laplace operator in $\mathbb{R}^3$. It is easy to see that Δ_{sph} is a second-order differential operator on the sphere; one can write explicit formulas for it in the spherical coordinates, but they are not particularly nice.

For many applications, it is important to know the eigenvalues and eigenfunctions of Δ_{sph}. In particular, this problem arises in quantum mechanics: the eigenvalues are related to the energy levels of a hydrogen atom in quantum mechanical description. Unfortunately, trying to find the eigenfunctions by brute force gives a second-order differential equation which is very difficult to solve.

However, it is easy to notice that this problem has some symmetry – namely, the group $\mathrm{SO}(3, \mathbb{R})$ acting on the sphere by rotations. How can one use this symmetry?

However, this will not solve the problem: for each individual rotation R, the eigenspaces will still be too large (in fact, infinite-dimensional), so diagonalizing Δ_{sph} in each of them is not very easy either. This is not surprising: after all, we only used one of many symmetries. Can we use all of rotations $R \in \mathrm{SO}(3, \mathbb{R})$ simultaneously?

This, however, presents two problems.

- $\mathrm{SO}(3, \mathbb{R})$ is not a finitely generated group, so apparently we will need to use infinitely (in fact uncountably) many different symmetries and diagonalize each of them.
- $\mathrm{SO}(3, \mathbb{R})$ is not commutative, so different operators from $\mathrm{SO}(3, \mathbb{R})$ can not be diagonalized simultaneously.

The goal of the theory of Lie groups is to give tools to deal with these (and similar) problems. In short, the answer to the first problem is that $\mathrm{SO}(3, \mathbb{R})$ is in a certain sense finitely generated – namely, it is generated by three generators, "infinitesimal rotations" around x, y, z axes (see details in Example 3.10).

The answer to the second problem is that instead of decomposing the $C^\infty(S^2, \mathbb{C})$ into a direct sum of common eigenspaces for operators $R \in \mathrm{SO}(3, \mathbb{R})$, we need to decompose it into "irreducible representations" of $\mathrm{SO}(3, \mathbb{R})$. In order to do this, we need to develop the theory of representations of $\mathrm{SO}(3, \mathbb{R})$. We will do this and complete the analysis of this example in Section 4.8.

Theorem 3.7. *Let G be a real or complex Lie group and $\mathfrak{g} = T_1G$.*

(1) $\exp(x) = 1 + x + \dots$ *(that is,* $\exp(0) = 1$ *and* $\exp_*(0)\colon \mathfrak{g} \to T_1G = \mathfrak{g}$ *is the identity map).*
(2) *The exponential map is a diffeomorphism (for complex G, invertible analytic map) between some neighborhood of* 0 *in* $\mathfrak{g}$ *and a neighborhood of* 1 *in G. The local inverse map will be denoted by* log.
(3) $\exp((t + s)x) = \exp(tx)\exp(sx)$ *for any* $s, t \in \mathbb{K}$.
(4) *For any morphism of Lie groups $\varphi\colon G_1 \to G_2$ and any $x \in \mathfrak{g}_1$, we have* $\exp(\varphi_*(x)) = \varphi(\exp(x))$.
(5) *For any $X \in G, y \in \mathfrak{g}$, we have* $X\exp(y)X^{-1} = \exp(\operatorname{Ad} X.y)$*, where* Ad *is the adjoint action of G on $\mathfrak{g}$ defined by* (2.4).

Proposition 3.9. *Let G_1, G_2 be Lie groups (real or complex). If G_1 is connected, then any Lie group morphism $\varphi\colon G_1 \to G_2$ is uniquely determined by the linear map $\varphi_*\colon T_1G_1 \to T_1G_2$.*

A basis in $\mathfrak{so}(3, \mathbb{R})$ is given by

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The corresponding one-parameter subgroups in $\mathrm{SO}(3, \mathbb{R})$ are rotations: $\exp(tJ_x)$ is rotation by angle t around x-axis, and similarly for y, z.

The commutation relations are given by

$$[J_x, J_y] = J_z, \qquad [J_y, J_z] = J_x, \qquad [J_z, J_x] = J_y. \tag{3.20}$$

A basis in $\mathfrak{su}(2)$ is given by so-called Pauli matrices multiplied by i:

$$\mathrm{i}\sigma_1 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \mathrm{i}\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathrm{i}\sigma_3 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}. \tag{3.21}$$

The commutation relations are given by

$$[\mathrm{i}\sigma_1, \mathrm{i}\sigma_2] = -2\mathrm{i}\sigma_3, \qquad [\mathrm{i}\sigma_2, \mathrm{i}\sigma_3] = -2\mathrm{i}\sigma_1, \qquad [\mathrm{i}\sigma_3, \mathrm{i}\sigma_1] = -2\mathrm{i}\sigma_2. \tag{3.22}$$

$\mathfrak{so}(3, \mathbb{R})$: elements J_x, J_y, J_z are orthogonal to each other, and $(J_x, J_x) = (J_y, J_y) = (J_z, J_z) = 2$

We can explicitly describe the corresponding subgroups in G. Namely,

$$\exp(tJ_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}$$

is rotation around x-axis by angle t; similarly, J_y, J_z generate rotations around y, z axes. The easiest way to show this is to note that such rotations do form a one-parameter subgroup; thus, they must be of the form $\exp(tJ)$ for some $J \in \mathfrak{so}(3, \mathbb{R})$, and then compute the derivative to find J.

By Theorem 3.7, elements of the form $\exp(tJ_x)$, $\exp(tJ_y)$, $\exp(tJ_z)$ generate a neighborhood of identity in $\mathrm{SO}(3, \mathbb{R})$. Since $\mathrm{SO}(3, \mathbb{R})$ is connected, by Corollary 2.10, these elements generate the whole group $\mathrm{SO}(3, \mathbb{R})$. For this reason, it is common to refer to J_x, J_y, J_z as "infinitesimal generators" of $\mathrm{SO}(3, \mathbb{R})$. Thus, in a certain sense $\mathrm{SO}(3, \mathbb{R})$ is generated by three elements.

Theorem 3.25. *Let G be a finite-dimensional Lie group acting on a manifold M, so we have a map $\rho: G \to \mathrm{Diff}(M)$. Then*

(1) *This action defines a linear map $\rho_* : \mathfrak{g} \to \mathrm{Vect}(M)$.*
(2) *The map ρ_* is a morphism of Lie algebras: $\rho_*[x, y] = [\rho_*(x), \rho_*(y)]$, where the commutator in the right-hand side is the commutator of vector fields.*

Theorem 3.29. *Let G be a Lie group acting on a manifold M (respectively, a complex Lie group holomorphically acting on a complex manifold M), and let $m \in M$.*

(1) *The stabilizer $G_m = \{g \in G \mid gm = m\}$ is a closed Lie subgroup in G, with Lie algebra $\mathfrak{h} = \{x \in \mathfrak{g} \mid \rho_*(x)(m) = 0\}$, where $\rho_*(x)$ is the vector field on M corresponding to x.*
(2) *The map $G/G_m \to M$ given by $g \mapsto g.m$ is an immersion. Thus, the orbit $\mathcal{O}_m = G \cdot m$ is an immersed submanifold in M, with tangent space $T_m\mathcal{O} = \mathfrak{g}/\mathfrak{h}$.*

We have an isomorphism of Lie algebras $\mathfrak{su}(2) \xrightarrow{\sim} \mathfrak{so}(3,\mathbb{R})$ given by

$$\begin{aligned} i\sigma_1 &\mapsto -2J_x \\ i\sigma_2 &\mapsto -2J_y \\ i\sigma_3 &\mapsto -2J_z. \end{aligned} \tag{3.25}$$

It can be lifted to a morphism of Lie groups $\mathrm{SU}(2) \to \mathrm{SO}(3,\mathbb{R})$, which is a twofold cover (see Exercise 2.8).

習作

Let J_x, J_y, J_z be the basis in $\mathfrak{so}(3,\mathbb{R})$ described in Section 3.10. The standard action of $\mathrm{SO}(3,\mathbb{R})$ on $\mathbb{R}^3$ defines an action of $\mathfrak{so}(3,\mathbb{R})$ by vector fields on $\mathbb{R}^3$. Abusing the language, we will use the same notation J_x, J_y, J_z for the corresponding vector fields on $\mathbb{R}^3$. Let $\Delta_{\mathrm{sph}} = J_x^2 + J_y^2 + J_z^2$; this is a second order differential operator on $\mathbb{R}^3$, which is usually called the *spherical Laplace operator*, or the *Laplace operator on the sphere*.

(1) Write Δ_{sph} in terms of $x, y, z, \partial_x, \partial_y, \partial_z$.
(2) Show that Δ_{sph} is well defined as a differential operator on a sphere $S^2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$, i.e., if f is a function on $\mathbb{R}^3$ then $(\Delta_{\mathrm{sph}} f)|_{S^2}$ only depends on $f|_{S^2}$.

(3) Show that the usual Laplace operator $\Delta = \partial_x^2+\partial_y^2+\partial_z^2$ can be written in the form $\Delta = \frac{1}{r^2}\Delta_{\mathrm{sph}} + \Delta_{\mathrm{radial}}$, where Δ_{radial} is a differential operator written in terms of $r = \sqrt{x^2+y^2+z^2}$ and $r\partial_r = x\partial_x + y\partial_y + z\partial_z$.

(4) Show that Δ_{sph} is rotation invariant: for any function f and $g \in \mathrm{SO}(3,\mathbb{R})$, $\Delta_{\mathrm{sph}}(gf) = g(\Delta_{\mathrm{sph}} f)$. (Later we will describe a better way of doing this.)

This problem is for readers familiar with the mathematical formalism of classical mechanics.
Let G be a real Lie group and A – a positive definite symmetric bilinear form on $\mathfrak{g}$; such a form can also be considered as a linear map $\mathfrak{g} \to \mathfrak{g}^*$.

(1) Let us extend A to a left invariant metric on G. Consider mechanical system describing free motion of a particle on G, with kinetic energy given by $A(\dot{g}, \dot{g})$ and zero potential energy. Show that equations of motion for this system are given by Euler's equations:

$$\dot{\Omega} = \mathrm{ad}^* v.\Omega$$

where $v = g^{-1}\dot{g} \in \mathfrak{g}$, $\Omega = Av \in \mathfrak{g}^*$, and ad^* is the coadjoint action:

$$\langle \mathrm{ad}^* x.f, y\rangle = -\langle f, \mathrm{ad}\, x.y\rangle \qquad x, y \in \mathfrak{g},\ f \in \mathfrak{g}^*.$$

(For $G = \mathrm{SO}(3, \mathbb{R})$, this system describes motion of a solid body rotating around its center of gravity – so called Euler's case of rotation of a solid body. In this case, A describes the body's moment of inertia, v is angular velocity, and Ω is angular momentum, both measured in the moving frame. Details can be found in [1]).

(2) Using the results of the previous part, show that if A is a bi-invariant metric on G, then one-parameter subgroups $\exp(tx), x \in \mathfrak{g}$ are geodesics for this metric.

Representation

Definition 4.1. A representation of a Lie group G is a vector space V together with a morphism $\rho: G \to \mathrm{GL}(V)$.

A representation of a Lie algebra $\mathfrak{g}$ is a vector space V together with a morphism $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$.

§6 參考書目

1. An Introduction to Lie Groups and Lie Algebras Alexander Kirillov
2. Lectures on Lie Groups 項武義
3. Foundations of Differential Manifolds and Lie Groups Frank W.Warner
4. Lie Algebras for Physicists Douglas W.McKenzie
5. Theory of Spinors
6. Lie Groups New Research Altos B. Canterra
7. Lie Groups，Differential Equations and Geometry Giovanni Falcone

Clifford Algebra and The Interpretation of Quantum Mechanics David Hestenes

Dirac Theory

ABSTRACT. The Dirac theory has a hidden geometric structure. This talk traces the conceptual steps taken to uncover that structure and points out significant implications for the interpretation of quantum mechanics. The unit imaginary in the Dirac equation is shown to represent the generator of rotations in a spacelike plane related to the spin. This implies a geometric interpretation for the generator of electromagnetic gauge transformations as well as for the entire electroweak gauge group of the Weinberg-Salam model. The geometric structure also helps to reveal closer connections to classical theory than hitherto suspected, including exact classical solutions of the Dirac equation.