§ Lie group and DE

Let X be a set. A one-to-one mapping φ of X onto X is called a *bijection*. Let $B(X)$ denote the set of all bijections of X onto X.

$$\frac{dy}{dx} = F(x, y)\,, \tag{3.1}$$

a *solution* being by definition a function $y = u(x)$ such that $u'(x) = F(x, u(x))$.

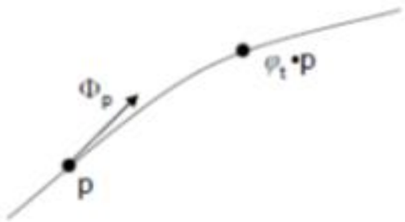
Thus a solution is a curve in $\mathbf{R}^2$ (an *integral curve*). A transformation $T \in B(X)$ $(X = \mathbf{R}^2)$ is said to leave the equation (3.1) *stable* if it permutes the integral curves.

先把(3.1)式寫成 $\frac{dy}{dx} = \frac{N(x,y)}{M(x,y)}$...(2)

and assume we have a 1-parameter group $\varphi_t (t \in \mathbf{R})$ of differentiable bijections of $\mathbf{R}^2$ leaving (3.2) stable.

Consider the *vector field* on $\mathbf{R}^2$

$$\Phi_p = \left\{ \frac{d(\varphi_t \cdot p)}{dt} \right\}_{t=0}$$



Here Φ_p is the tangent vector to the orbit $\varphi_t \cdot p$ at $p \in \mathbf{R}^2$. Thinking of a vector at $p = (x, y)$ as a directional derivative we write the vector field in the form

$$\Phi_p = \Phi_{x,y} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}\,. \tag{3.3}$$

定理(Lie 1874)

假設 φ_t 是(2)穩定(stable)的單參數群(1-parameter group)，則存在函數 U(x,y)使得

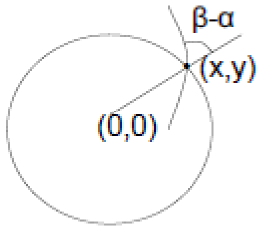
$\frac{\partial U}{\partial x} = \frac{-N}{M\eta - N\xi}, \frac{\partial U}{\partial y} = \frac{M}{M\eta - N\xi}$ ，且 U(x,y)=const.是(2)式的解

即 $\frac{Mdy - Ndx}{M\eta - N\xi} = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy$ 是一個 exact form

而 $(M\eta - N\xi)^{-1}$ 是 Mdy-Ndx=0 的積分因子

例 $\frac{dy}{dx}=\frac{y+x(x^2+y^2)}{x-y(x^2+y^2)}$

(3.6) $$\frac{\frac{dy}{dx}-\frac{y}{x}}{1+\frac{y}{x}\frac{dy}{dx}}=x^2+y^2$$



The slope of the ray from $(0,0)$ to (x,y) is $\frac{y}{x}=\tan\alpha$ and the slope of the tangent to the integral curve through (x,y) is $\frac{dy}{dx}=\tan\beta$. Since

$$\tan(\beta-\alpha)=\frac{\tan\beta-\tan\alpha}{1+\tan\alpha\tan\beta}$$

(3.6) states that
$$\tan(\beta-\alpha)=x^2+y^2\,.$$

This means that the angle $\beta-\alpha$ is constant as (x,y) varies on a circle with center $(0,0)$. Thus each rotation

(3.7) $$\varphi_t:(x,y)\to(x\cos t-y\sin t\,,\,x\sin t+y\cos t)$$

maps each integral curve into another integral curve, in other words leaves the equation (3.5) stable. Also the rotations φ_t form a group $(\varphi_{t+s}=\varphi_t\varphi_s)$. Here we have from (3.7)

$$\Phi_p=\left(\frac{d(\varphi_t\cdot p)}{dt}\right)_{t=0}=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}$$

so by Lie's theorem

$$\left[(x-y(x^2+y^2))x-(y+x(x^2+y^2))(-y)\right]^{-1}=(x^2+y^2)^{-1}$$

is an integrating factor. Also

$$\frac{Mdy-Ndx}{x^2+y^2}=(\frac{x}{x^2+y^2}-y)dy-(\frac{y}{x^2+y^2}+x)dx\text{ 即 }\frac{\partial U}{\partial x}dx+\frac{\partial U}{\partial y}dy$$

$$\hat{x}=x\cos t-y\sin t,\hat{y}=x\sin t+y\cos t$$

$$(\xi,\eta)=\frac{d}{dt}(\hat{x},\hat{y})|_{t=0}=(-y,x)$$

所以 $X=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}$

$$(\frac{xdy - ydx}{x^2 + y^2}) - (ydy + xdx) = 0$$

$U(x, y) = \arctan(\frac{y}{x}) - \frac{x^2 + y^2}{2}$，解 U(x,y)=c 可以寫成 $y = x\tan(\frac{1}{2}(x^2 + y^2) + c)$

註 $\frac{xdy - ydx}{x^2 + y^2} = d(\tan^{-1}\frac{y}{x})$

5 Proof of Lie's Theorem

So far as I know this interesting theorem does not occur in most recent books on ordinary differential equations. Older proofs seem a bit obscure (but take a look at Lie's original proof (Collected Works, Vol. 3)). The proof in Olver's book is clean and rigorous but contained in a longer theory of prolongations.

Below is a short proof. Suppose φ_t is a 1-parameter group leaving the equation

$$\frac{dy}{dx} = \frac{Y(x, y)}{X(x, y)} \tag{5.1}$$

stable. If $U(x, y) = c$ is a solution we have with $\varphi_t(x, y) = (x_t, y_t)$,

$$U(x_t, y_t) = c(t) \qquad \text{(all } t\text{)}$$

so

$$\frac{\partial U}{\partial x}\frac{dx_t}{dt} + \frac{\partial U}{\partial y}\frac{dy_t}{dt} = c'(t)$$

and by (3.3)

$$\frac{\partial U}{\partial x}\xi + \frac{\partial U}{\partial y}\eta = c'(0)\,. \tag{5.2}$$

Secondly

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y}\frac{dy}{dx} = 0$$

so

$$\frac{\partial U}{\partial x}X + \frac{\partial U}{\partial y}Y = 0\,. \tag{5.3}$$

If $c'(0) \neq 0$ we can normalize U such that $c'(0) = 1$. Then (5.2) and (5.3) imply

$$\frac{\partial U}{\partial x} = \frac{-Y}{X\eta - Y\xi}\,, \quad \frac{\partial U}{\partial y} = \frac{X}{X\eta - Y\xi}$$

so $(X\eta - Y\xi)^{-1}$ is an integrating factor for $X\,dy - Y\,dx = 0$ as claimed.

On the other hand, if $c'(0) = 0$, (5.2)–(5.3) imply $dy/dx = \eta/\xi$ so the integral curves are just the orbits of φ_t.

形如 $\frac{dy}{dx} = f(\frac{y}{x})$ 的李群是 $\varphi_t : (x, y) = (e^t x, e^t y)$

所以 $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ 積分因子為 $(y - f(\frac{y}{x})x)^{-1}$