§ SO(2)到 SO(3)

李群在物理上的應用

(1) 角動量在量子力學中行為如何

(2) 基本粒子

(3) 規範場論中的對稱性與不變量

最終是希望研究 SO(3) 原因有二

對李群與李代數的抽象概念找一個具體的例子 看它如何運作。

一個 Laplace 算子的固有值(eigenvalues)與氫原子的能階有關。

這個 Laplace 算子是一個難解的二階微分方程 而 SO(3)對此有些幫助 其中當然還有困難 還需要研究 SO(3)的表現理論。

SU(2) locally isomorphic to SO(3) 但是可以推到更高維的 Lie algebra 所以更形重要。

SO(2) 平面的旋轉群：通常以 $A(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$表現

A(θ) 在 θ=0 展開= $I + A'(0)\theta + \frac{A''(0)}{2!}\theta^2 + \frac{A'''(0)}{3!}\theta^3 + ...$

$\theta \approx \varepsilon$, $A(\varepsilon) \approx I + A'(0)\varepsilon$ 令 $A(\varepsilon) \approx I + i\varepsilon X$,其中 $X = -iA'(0)$

$$X = -iA'(0) = -i\frac{d}{d\theta}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\Bigg|_{\theta=0} = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}$$

$$e^{i\theta X} = I + i\theta X + \frac{(i\theta X)^2}{2!} + \frac{(i\theta X)^3}{3!} + \frac{(i\theta X)^4}{4!} + ...$$

其中 $iX = \begin{pmatrix}o & -1\\ 1 & 0\end{pmatrix}$ ，$(iX)^2 = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} = -I$

= $I(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-..) + iX(\theta - \frac{\theta^3}{3!}+...) = I\cdot\cos\theta + iX\sin\theta$

= $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\cos\theta + \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\sin\theta = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$

其中 $X = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}$稱為 SO(2)的 Lie Algebra s0(2)的 generator，包含了 SO(2)的所有性質。

總結：

A generator of the algebra of a one parameter Lie group, SO(2), was derived. The generator was derived from a particular representation of the group which expressed the group structure naturally,

Exponentiating the generator was shown to recreate the starting representation of the group which demonstrates that the generator contains all of the information of the group structure.

Other starting representations were used and the resulting generators, when exponentiated, yielding representations which were not the same as the starting representation.

SO(3)：A Lie group with three parameters $\theta_1, \theta_2, \theta_3$

The group of rotations in three dimensions

$$M_1(\theta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix} \quad M_2(\theta_2) = \begin{pmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}$$

$$M_3(\theta_3) = \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

三個 generators

$$X_1 = -i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} 、 X_2 = -i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} 、 X_3 = -i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$[X_i, X_j] = iC_{ij}^k X_k$ ，其中 C_{ij}^k 稱為此 Lie algebra 的結構常數。

$$O(3) = \{A \in M \mid A^t A = I\}$$

$SO(3) = \{A \in \mathrm{O}(3) \mid \det A = 1\}$ subgroup of O(3)　dim=3

$$\mathfrak{g} = \{A \mid A + A^t = O\}$$

$U(n) = \{A \in M_{n\times n}(C) \mid A^t A = I\}$ unitary group ，$SU(n) = \{A \in U(n) \mid \det A = 1\}$

通過指數映射，也就是 Rodrigues' Formula(Ivory-Jacobi formula)，我們能把 so(3) 中的旋轉向量對應至 SO(3) 中的一個旋轉矩陣。反之則能夠使用對數映射將 SO(3) 的旋轉矩陣對應至 so(3) 中的旋轉向量。

We could use the SO(3) group to analyze the angular momentum of multi-particle systems in three space but there is another group, SU(2), which is locally isomorphic to SO(3), that we will use instead. SU(2) is not only important for studying rotations but it is also important when studying higher dimensional Lie algebras. We said before that we would use the Lie algebra for SO(3) to analyze angular momentum of multi-particle systems. Now we are saying that we are going to use SU(2) rather than SO(3) – what's going on? It turns out that SU(2) is so similar to SO(3) that they have identical Lie algebras, i.e.

$$[X_i, X_j] = i\varepsilon_{ijk}X_k$$

We choose to use SU(2) because it is so important in higher dimensional Lie algebras and this is a perfect opportunity to work with it. SU(2) is the Special (determinant =1), Unitary

https://blog.csdn.net/shao918516/article/details/116604377

[Lie Algebras for Physicists] by Robert Klauber