§ Spin group and Spin structure

Spin(n) is the non-trivial 2-fold covering of the special orthogonal group SO(n)。

It is a Lie group，connected if $n \geq 2$ and simple connected if $n \geq 3$。

If $Spin(n) \xrightarrow{\xi} SO(n)$ then $\xi(z) = z^2$ for any $z \in Spin(2) \cong S^1 = \{z \in C \| z| = 1\}$

例 Spin(4)

$Spin(4) \cong SU(2) \times SU(2)$

Lie algebra $spin(4) \cong su(2) \oplus su(2)$ 在 4 維時，旋量自然分成 Weyl spinor。

Ex

Compute $\exp(\alpha e_{12} + \beta e_{34})$

因為 $(e_{12})^2 = -1$ $\exp(\alpha e_{12}) = \cos\alpha + e_{12}\sin\alpha$

$$\exp(\alpha e_{12} + \beta e_{34}) = (\cos\alpha + e_{12}\sin\alpha)(\cos\beta + e_{34}\sin\beta)$$
$$= \cos\alpha\cos\beta + e_{12}\sin\alpha\cos\beta + e_{34}\cos\alpha\sin\beta + e_{1234}\sin\alpha\sin\beta$$

$$u = \langle u \rangle_0 + \langle u \rangle_1 + \langle u \rangle_2 + \langle u \rangle_3 + \langle u \rangle_4$$

$$\tilde{u} = \langle u \rangle_0 + \langle u \rangle_1 - \langle u \rangle_2 - \langle u \rangle_3 + \langle u \rangle_4$$ (reversion 反轉)

$$Spin(4) = \{s \in cl_4^+ \mid s\tilde{s} = 1\}$$

What is the cl_4^+ ?

Even subalgebra of Clifford algebra generated by $\{1, e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}, e_{1234}\}$

Spin(4)：

The most common way to define Spin(4) is as the universal double cover of the special orthogonal group SO(4)。

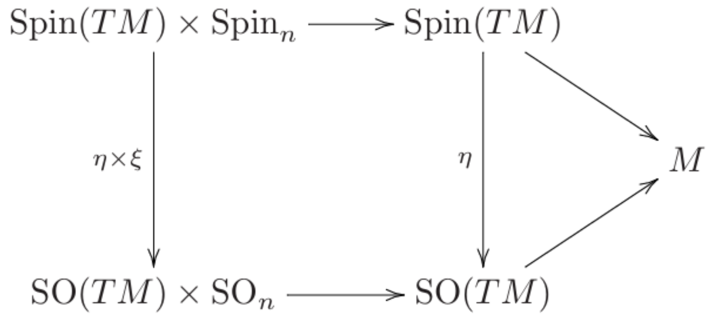
SO(4)：This is the group of all rotations in 4-dimensional Euclidean space。Just like SO(3) describes rotations in 3D，SO(4) describes how you can rotate objects in 4D space around a fixed origin。

Double Cover：This means that for every rotation in SO(4)，there are exactly two corresponding elements in Spin(4)。

Definition

(1) A spin structure on an oriented Riemannian manifold (M,g) is a Spin(n)-principal bundle $Spin(TM) \to M$ together with a 2-fold covering map $Spin(TM) \xrightarrow{\eta} SO(TM)$ compatible with the respective group action，i.e. the following

diagram commute：



(2) A spin manifold is an oriented Riemannian manifold admitting a spin structure。

A spin structure：

(1) Manifold

(2) Frame bundle：At each point on the manifold，you can imagine a set of orthonormal basis vectors (a "frame")。
The collection of all possible frames at all points is the frame bundle。
The structure of these frames is described by the rotation group SO(n)。

(3) Spin(n) group

(4) Lifting：A spin structure is a way of consistently choosing one of the two Spin(n) elements for every SO(n) rotation of your frames across the entire manifold。