§ Local theory of Symplectic manifolds

2.1 Isotopy

Definition

Let M is a manifold，$\phi : M \times R \to M$

If $\phi_t : M \to M$ satisfies

1. $\phi_0 = id$
2. $\phi_t(m) := \phi(m,t)$ is a diffeomorphism for every $t \in R$

Then ϕ is an isotopy

For every isotopy ϕ，we construct a vector field X_t on M by letting

$\dfrac{d\phi_t}{dt} = X_t \circ \phi_t$ or，in other words，$\phi_*(\dfrac{\partial}{\partial t}\Big|_{(m,t)}) = X_{t \circ \phi_t(m)}$

Where $\phi_* : T_{(m,t)}(M \times R) \to T_{\phi_t(m)}M$ is the push-forward of ϕ。

Hence there is a family of vector fields $\{X_t\}$ in M。

On the other hand，given a R-famiy of compactly-supported vector fields $\{X_t\}$，we can solve the differential equation $\dfrac{d\phi_t}{dt} = X_t \circ \phi_t$ to get back the isotopy ϕ。

Definition

A one-parameter group of diffeomorphisms of a manifold is an isotopy with the extra property $\phi_{s+t} = \phi_s \circ \phi_t$

例 X is a vector field on a manifold M，$\varphi_t : M \to M$

1. $\varphi_0(m) = m$ for $\forall m \in M$
2. $\dfrac{d\varphi_t(m)}{dt} = X_{(\varphi_t(m))}$

Then φ_t is the exponential map (or the flow) of X，denote φ_t by exp(t X)

Definition

The Lie derivative of a form α along a vector field X is given by

$$L_X \alpha := \frac{d}{dt}\varphi_t^* \omega\Big|_{t=0}$$

Cartan formula $L_X \omega = d(i_x \omega) + i_X d\omega$

Proposition

For a family of 2-form ω_t ， $\frac{d}{dt}\varphi_t^*\omega_t = \varphi_t^*(L_{X_t}\omega_t + \frac{d\omega_t}{dt})$

Proof

$\frac{d}{dt}\varphi_t^*\omega = \varphi_t^* L_X \omega$ ，then

$$\frac{d}{dt}\varphi_t^*\omega_t = (\frac{d}{ds}\varphi_s^*)\Big|_{s=t\omega_t} + \varphi_t^*(\frac{d}{ds}\omega_s)\Big|_{s=t}$$

$$= \varphi_t^* L_{X_t}\omega_t + \varphi_t^*\frac{d\omega_t}{dt} = \varphi_t^*(L_{X_t}\omega_t + \frac{d\omega_t}{dt})$$

2.2 Moser Theorem

Let M be a compact manifold，and $\omega_0, \omega_1 \in \Omega^2(M)$ be in same de Rham cohomology (餘調)group。Suppose $\omega_t = (1-t)\omega_0 + t\omega_1$ be symplectic for all $t \in [0,1]$ ，then there is an isotopy ϕ such that $\phi_t^*\omega_t = \omega_0$ for all $t \in [0,1]$

($\omega_0, \omega_1 \in \Omega^2(M)$ be in same de Rham cohomology group $\Leftrightarrow$ $\omega_0 - \omega_1$ is exact。i.e.

$\exists \sigma \in \Omega^1$ ，such that $\omega_1 - \omega_0 = d\sigma$)

Proof

$\omega_t = (1-t)\omega_0 + t\omega_1$ ， $\therefore \frac{d\omega_t}{dt} = \omega_1 - \omega_0 = d\sigma$

From Cartan formula $L_{X_t}\omega_t = i_{X_t}d\omega_t + d(i_{X_t}\omega) = d(i_{X_t}\omega)$

又 $\frac{d}{dt}\phi_t^*\omega_t = \phi_t^*(L_{X_t}\omega_t + \frac{d\omega_t}{dt}) = \phi_t^*(d(i_{X_t}\omega_t) + d\sigma))$

If $d(i_{X_t}\omega_t) + d\sigma = 0$ then $\frac{d}{dt}\phi_t^*\omega_t = 0 \Rightarrow \phi_t^*\omega_t = c$

there is an isotopy ϕ such that $\phi_t^*\omega_t = \omega_0$ for all $t \in [0,1]$

Moser's equation : $i_{X_t}\omega_t + \sigma = 0$

But ω_t is non-degenerate, so we can solve X_t for each $t \in [0,1]$ smoothly by the uniqueness theorem of differential equations. Given such X_t we can find its isotopy by compactness of M. □

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Theorem 2.8 (Tubular Neighbourhood Theorem). *Suppose Q is an submanifold of a manifold M, the normal bundle of Q is defined by*

$$NQ = \{(q,n) : q \in X; n \in N_xQ := \frac{T_xM}{T_xQ}\}$$

Then there exist a convex neighbourhood $\tilde{\mathcal{U}}$ of the zero section of NQ, a neighbourhood $\mathcal{U}$ of Q, and a diffeomorphism $\varphi : \tilde{\mathcal{U}} \to \mathcal{U}$ such that $\phi(q,0) = q$ for all $q \in Q$.

2.3 Darboux Theorem

Every symplectic form ω of a 2n-dimenional symplectic manifold M is locally diffeomorphic to the standard form

$\sum_{i=1}^{n} dx_i \wedge dy_i$ on R^{2n}

M 為 2n 維光滑流形，ω 為在點 $x \in M$ 鄰域的非退化 閉 2-form，則在 x 鄰域可選局部座標系 $\{q^1, q^2, ..., q^n, p_1, ..., p_n\}$ 使得 $\omega = \sum_i dp_i \wedge dq^i$ 。

此局部座標稱為 Darboux 座標。