§ 幾何力學

[DG01]黎曼幾何及其在力學與相對論的應用

Definition 1.2 A **mechanical system** is a triple $(M, \langle\cdot,\cdot\rangle, \mathcal{F})$, where:

(i) M is a differentiable manifold, called the **configuration space**;

(ii) $\langle\cdot,\cdot\rangle$ is a Riemannian metric on M yielding the **mass operator** $\mu : TM \to T^*M$, defined by

$$\mu(v)(w) = \langle v, w\rangle$$

for all $v, w \in T_pM$ and $p \in M$;

(iii) $\mathcal{F} : TM \to T^*M$ is a differentiable map satisfying $\mathcal{F}(T_pM) \subset T_p^*M$ for all $p \in M$, called the **external force**.

A **motion** of the mechanical system is a solution $c : I \subset \mathbb{R} \to M$ of the **Newton equation**

$$\mu\left(\frac{D\dot{c}}{dt}\right) = \mathcal{F}(\dot{c}).$$

Definition 4.1 A **distribution** Σ of dimension m on a differentiable manifold M is a choice of an m-dimensional subspace $\Sigma_p \subset T_pM$ for each $p \in M$. The distribution is said to be **differentiable** if for all $p \in M$ there exists a neighborhood $U \ni p$ and vector fields $X_1, \ldots, X_m \in \mathfrak{X}(U)$ such that

$$\Sigma_q = \operatorname{span}\left\{(X_1)_q, \ldots, (X_m)_q\right\}$$

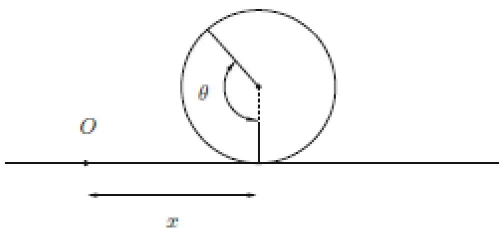
for all $q \in U$.

Equivalently, Σ is differentiable if for all $p \in M$ there exists a neighborhood $U \ni p$ and 1-forms $\omega^1, \ldots, \omega^{n-m} \in \Omega^1(U)$ such that

$$\Sigma_q = \ker\left(\omega^1\right)_q \cap \cdots \cap \ker\left(\omega^{n-m}\right)_q$$

for all $q \in U$

例 1. Wheel rolling without slipping



$\dot{x} = R\dot{\theta}$

Configuration space is $R \times S^1$

$X = R\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial \theta}$ ，the kernel of the 1-form $\omega = dx - Rd\theta$

$d\omega = 0$ this is a semi-holonomic constraint，corresponding to an integrable distribution.

The leaves of the distribution are the submanifolds with equation $x = x_0 + R\theta$

Since $\mu^{-1}\mathcal{R}$ is orthogonal to the constraint for a perfect reaction force $\mathcal{R}$, the constraint must be in the kernel of $\mathcal{R}$, and hence $\mathcal{R} = \lambda\omega$ for some smooth function $\lambda : TM \to \mathbb{R}$.

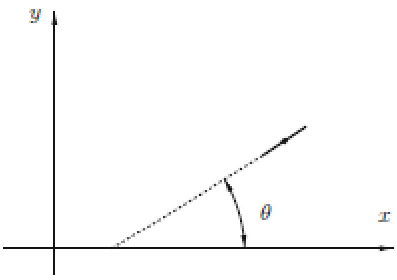
If the kinetic energy of the wheel is

$$K = \frac{M}{2}(v^x)^2 + \frac{I}{2}(v^\theta)^2$$

then

$$\mu\left(\frac{D\dot{c}}{dt}\right) = M\ddot{x}dx + I\ddot{\theta}d\theta.$$

例 2.Ice skate



Configuration space is $R^2 \times S^1$

$X = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$，$Y = \frac{\partial}{\partial\theta}$

The kernel of the 1-form $\omega = -\sin\theta dx + \cos\theta dy$

$d\omega \wedge \omega = -d\theta \wedge dx \wedge dy \neq 0$

This a true non-holonomic constraint.

(a) Show that the ice skate can access all points in the configuration space: given two points $p, q \in \mathbb{R}^2 \times S^1$ there exists a piecewise smooth curve $c : [0, 1] \to \mathbb{R}^2 \times S^1$ compatible with Σ such that $c(0) = p$ and $c(1) = q$. Why does this show that Σ is non-integrable?

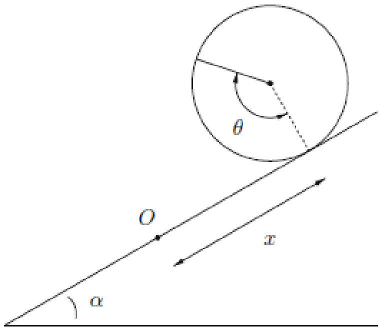
(b) Assuming that the kinetic energy of the skate is

$$K = \frac{M}{2}\left((v^x)^2 + (v^y)^2\right) + \frac{I}{2}(v^\theta)^2$$

and that the reaction force is perfect, show that the skate moves with constant speed along straight lines or circles. What is the physical interpretation of the reaction force?

(c) Determine the motion of the skate moving on an inclined plane, i.e. subject to a potential energy $U = Mg\sin\alpha\, x$.

例 3.



Wheel rolling without slipping on an inclined plane

$$\begin{cases} M\ddot{x} = -Mg\sin\alpha + \lambda \\ I\ddot{\theta} = -R\lambda \\ \dot{x} = R\dot{\theta} \end{cases} \Rightarrow \begin{cases} x(t) = x_0 + v_0 t - \dfrac{\gamma}{2}t^2 \\ \theta(t) = \theta_0 + \dfrac{v_0}{R}t - \dfrac{\gamma}{2R}t^2 \\ \lambda = \dfrac{I\gamma}{R^2} \end{cases}$$，where $\gamma = \dfrac{g\sin\alpha}{1 + \dfrac{I}{MR^2}}$

雙擺的 Lagrangian

$$L = \frac{1}{2}(m_1 + m_2)l_1^2\dot{\theta}^2 + \frac{1}{2}m_2(l_2^2\dot{\varphi}^2 + 2l_1l_2\dot{\theta}\dot{\varphi}\cos(\theta - \varphi) + m_1gl_1\cos\theta + m_2g(l_1\cos\theta + l_2\cos\varphi)$$

運動方程式為

$$\phi(\theta,\varphi) = (l_1\sin\theta, -l_1\cos\theta, l_1\sin\theta + l_2\sin\varphi, -l_1\cos\theta - l_2\cos\varphi)$$

Lemma 1.4.4 *The Euler–Lagrange equations for the energy E are*

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0, \quad i = 1, \ldots, d \tag{1.4.15}$$

Proof The Euler–Lagrange equations of a functional

$$I(x) = \int_a^b f(t, x(t), \dot{x}(t))dt$$

are given by

$$\frac{d}{dt}\frac{\partial f}{\partial \dot{x}^i} - \frac{\partial f}{\partial x^i} = 0, \quad i = 1, \ldots, d.$$

In our case, recalling

$$E(\gamma) = \frac{1}{2}\int g_{jk}(x(t))\dot{x}^j\dot{x}^k dt,$$

we get

$$\frac{d}{dt}\left(g_{ik}(x(t))\dot{x}^k(t) + g_{ji}(x(t))\dot{x}^j(t)\right) - g_{jk,i}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0,$$

for $i = 1, \ldots, d$, hence

$$g_{ik}\ddot{x}^k + g_{ji}\ddot{x}^j + g_{ik,\ell}\dot{x}^\ell\dot{x}^k + g_{ji,\ell}\dot{x}^\ell\dot{x}^j - g_{jk,i}\dot{x}^j\dot{x}^k = 0.$$

Renaming some indices and using the symmetry $g_{ik} = g_{ki}$, we get

$$2g_{\ell m}\ddot{x}^m + (g_{\ell k,j} + g_{j\ell,k} - g_{jk,\ell})\dot{x}^j\dot{x}^k = 0, \quad \ell = 1, \ldots, d, \tag{1.4.16}$$

and from this

$$g^{i\ell}g_{\ell m}\ddot{x}^m + \frac{1}{2}g^{i\ell}(g_{\ell k,j} + g_{j\ell,k} - g_{jk,\ell})\dot{x}^j\dot{x}^k = 0, i = 1, \ldots, d.$$

Because of

$$g^{i\ell}g_{\ell m} = \delta_{im}, \text{ and thus } g^{i\ell}g_{\ell m}\ddot{x}^m = \ddot{x}^i,$$

we obtain (1.4.15) from this. □

Definition 1.4.2 A smooth curve $\gamma = [a, b] \to M$, which satisfies (with $\dot{x}^i(t) = \frac{d}{dt}x^i(\gamma(t))$ etc.)

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0, \text{ for } i = 1, \ldots, d \tag{1.4.17}$$

is called a *geodesic*.

Thus ，geodesics are the critical points of the energy functional。