§ Completely Integrable Systems

$H \in C^{\infty}(T^{*}M)$ is a Hamiltonian function，being constant along its Hamiltonian flow。

Definition

A function $F \in C^{\infty}(T^{*}M)$ is said to be a first integral of H if {H ,F}=0

Definition

The functions $F_1, F_2, ..., F_n \in C^{\infty}(T^{*}M)$ are said to be

1. In involution if $\{F_i, F_j\} = 0$

2. Independent at $\alpha \in T^{*}M$ if $(dF_1)_\alpha, (dF_2)_\alpha, ..., (dF_n)_\alpha \in T^{*}_\alpha(T^{*}M)$ are linearly independent convectors

Proposition

If $F_1, F_2, ..., F_n \in C^{\infty}(T^{*}M)$ are in involution and are independent at some point $\alpha \in T^{*}M$ then $m \le n$

Definition

The Hamiltonian H is said to be completely integrable if there exist n first integrals $F_1, F_2, ..., F_n$ in involution which are independent on a dense open set $U \subset T^{*}M$

Proposition

Let H be a completely integrable Hamiltonian with first integrals $F_1, F_2, ..., F_n$ in involution，independent in the dense open set $U \subset T^{*}M$，and such that $X_{F_1}, X_{F_2}, ..., X_{F_n}$ are complete on U，then each nonempty level set

$L_f := \{\alpha \in U \mid F_i(\alpha) = f_i\}$ is a submanifold of dimension n，invariant for the Hamiltonian H flow of H，admitting a locally free action of R^n which is transitive on each connected component。

後面還有兩個命題 狀況外

定理 Arnold-Liouville

Theorem 7.10 (Arnold–Liouville) *Let H be a completely integrable Hamiltonian with n first integrals $F_1, \dots, F_n \in C^\infty(T^*M)$ in involution, independent on the dense open set $U \subset T^*M$. If the connected components of the level sets of the map $(F_1, \dots, F_n) : U \to \mathbb{R}^n$ are compact then they are n-dimensional tori, invariant for the flow of X_H. The flow of X_H on these tori is a linear flow (for an appropriate choice of coordinates).*