§ symplectic forms

A symplectic form ω on a vector space V

$\omega : V \times V \to R$

1. Skew-symmetric
2. Non-degenerate if $\omega(v,w)=0$ for $\forall w \in V$ then v=0

And (V,ω) is called a symplectic space

There exists a basis $e_1, e_2, ..., e_n, f_1, ..., f_n$ such that

$\omega(e_i, e_i) = \omega(f_j, f_j) = 0$ and $\omega(e_i, f_j) = \delta_{ij}$

Definition

Let (V,ω) be a symplectic space，then for any subspace W of V

1. W is isotropic if $\omega(w_1, w_2) = 0$ for $\forall w_1, w_2 \in W$

   Denote W^{ω} by $W^{\omega} := \{v \in V : \omega(w,v) = 0, \forall w \in W\}$，it is easy to see that W is isotropic $\Leftrightarrow W \subseteq W^{\omega}$
2. W is Lagrangian if W is an isotropic space of maximal dimension

Definition

Let $(V,\omega),(V',\omega')$ be symplectic vector space。

$\phi : V \to V'$ such that $\omega'(\phi(u),\phi(v)) = \omega(u,v), \forall u,v \in V$ is a linear isomorphism

In other words $\phi^{*}\omega' = \omega$

Then we say $(V,\omega),(V',\omega')$ are symplectomorphic

Definition

A symplectic form on a manifold is a differential 2-form ω on M such that

1. $\omega_p : T_pM \times T_pM \to R$ is symplectic $\forall p \in M$

   在流形上取局部座標系 $\omega = \frac{1}{2}\omega_{ij} dx^i \wedge dx^j$

   non-degenerate $\det(\omega_{ij}) \neq 0$

2. ω is closed

Then (M,ω) is a symplectic manifold

稱ω為 M 上的辛結構。

例 10.1

相位空間(configuration space)N 的餘切叢 $M = T^*N$ 對於物理應用而言是最重要的。

在 T^*N 座標系 $\{p_i, q^j\}$ 引入一 regular 1-form $\vartheta = \sum_i p_i dq^i \in \Lambda^1(M)$

則 $\omega = d\vartheta = \sum_i dp_i \wedge dq^i \in \Lambda^2(M)$ 給出 M 上的辛結構。

ϑ 稱為 symplectic potential。

在經典力學中 N 稱為相位空間，M 稱為哈密頓體系的相空間。

$\{p_i\}$ 為廣義座標，$\{q^i\}$ 為對偶的廣義動量。

例 10.2 李群的餘伴隨軌道 p.310 侯

例 10.3 複投射流形 Kahler 流形

例 A particle (x_1, x_2, x_3) moving in R^3 with momentu $M = T^*N$ m (p_1, p_2, p_3)。

Suppose the energy function $H(x,p) = \frac{1}{2m}|p|^2 + U(x)$ ，where U(x) is the potential

energy function satisfying $m\frac{d^2x}{dt^2} = -\nabla U(x)$

Then we have the following Hamilton equations

$$\begin{cases} \dfrac{dx_i}{dt} = \dfrac{\partial H}{\partial p_i} \\ \dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial x_i} \end{cases}$$

In classical mechanics，solving Hamilton equations is equivalent to solving the Euler-Lagrange equation of the Lagrange L。

Let our symplectic manifold be R^6 with coordinates $(x_1, x_2, x_3, p_1, p_2, p_3)$

Then $\omega = \sum_i dx_i \wedge dp_i$

$\alpha \in T^*M$ ，$\pi(x^i, p_i) = (x^i)$

$$v = \sum_i dx^i(v)\frac{\partial}{\partial x^i} + \sum_i dp_i \frac{\partial}{\partial p_i} \in T_\alpha(T^*M)$$

$\theta \in \Omega^1(T^*M)$ is a 1-form given by $\theta_\alpha(v) := \alpha((d\pi)_\alpha(v))$

Then $(d\pi)_\alpha(v) = \sum_i dx^i(v)\frac{\partial}{\partial x^i}$

$\theta_\alpha(v) = \alpha((d\pi)_\alpha(v)) = \sum_i p_i dx^i(\sum_j dx^j(v)\frac{\partial}{\partial x^j}) = \sum_i p_i dx^i(v)$

$\theta = \sum_i p_i dx^i$ is a regular 1-form

$\omega = d\theta = \sum_i dp_i \wedge dx^i$ is a 2-form called canonical symplectic form on T^*M

(註:若取 local coordinates 為$\{p_i, q^i\}_{i=1}^n$ 則$\theta = \sum_i p_i dq^i, \omega = d\theta = \sum_i dp_i \wedge dq^i$)

Proposition

The canonical symplectic form ω is closed and nondegenerate。

Moreover $\omega^n = \omega \wedge ... \wedge \omega$ is a volume form。

由於ω為非退化，故可在切場與餘切場之間建立一對映關係

$\mu : \chi(M) \to \Lambda^1(M)$

$X = \xi^i \frac{\partial}{\partial x^i} \to -i_X\omega = -\omega_{ij}\xi^i dx^j$

i.e. $\mu(X) = -i_X\omega = -\omega_{ij}\xi^i dx^j$

Proposition

The Hamilton equations are the equations for the flow of the vector field X_H satisfying $i(X_H)\omega = -dH$

Proof

The Hamilton equations yield the flow of the vector field

$X_H = \sum_i(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i})$ ， $\omega = d\theta = \sum_i dp_i \wedge dx^i$

Therefore

$$i(X_H)\omega = i(X_H)\sum_i(dp_i \otimes dx^i - dx^i \otimes dp_i)$$

$$= \sum_i(-\frac{\partial H}{\partial x^i}dx^i - \frac{\partial H}{\partial p_i}dp_i) = -dH$$

Definition

The Hamiltonian flow generated by $F \in C^{\infty}(T^*M)$ is the flow of the unique vector field $X_F \in \chi(T^*M)$ such that $i(X_F)\omega = -dF$

Proposition

Hamiltonian flows preserve their generating functions。i.e. $X_F F = 0$

Proof

$X_F F = dF(X_F) = (-i(X_F)\omega)(X_F) = -\omega(X_F, X_F) = 0$ as ω is alternating。

Proposition

Hamiltonian flows preserve the canonical symplectic form。

If $\varphi_t : T^*M \to T^*M$ is a Hamiltonian flow then $\varphi^*\omega = \omega$

換句話說 在辛流形(M,ω)上 保辛結構變換的無窮小生成元$X \in \chi(M)$稱為辛向量場，滿足$L_X\omega = 0$ ($di_X\omega = 0$ i.e. $i_X\omega$ is closed。)

假設 $-i_X\omega = df$ ，

注意到辛结构 ω 为非退化 2 形式,(10.4)式映射 μ 为 1－1 对应的实线性映射,存在逆映射 μ^{-1},对于流形上任意光滑函数 $f \in F(M)$,映射

$$\mu^{-1}: \quad \mathrm{d}f \in \Lambda^1(M) \to X_f \in \mathfrak{X}(M) \tag{10.9}$$

向量场 $X_f = \mu^{-1}(\mathrm{d}f)$称为哈密顿矢场,满足

$$-i_{X_f}\omega = \mathrm{d}f \tag{10.9a}$$

我们知道正合形式必是闭形式,反之不一定.对应地,哈密顿矢场必是辛矢场,反之不一定.用 Ham(M)表示辛流形上所有哈密顿矢场集合,以上分析表明

$$\mathrm{Ham}(M) \subset \mathrm{Sym}(M) \subset \mathfrak{X}(M)$$

哈密頓向量場一定是辛向量場，反之 不一定成立。

Liouville theorem

Hamiltonian flows preserve the integral with respect to the symplectic volume form。

$\varphi_t : T^*M \to T^*M$ is a Hamiltonian flow and $F \in C^{\infty}(T^*M)$ is a compactly supported function then $\int\limits_{T^*M} F \circ \varphi_t = \int\limits_{T^*M} F$

Proof

$\varphi_t^*\omega = \omega$

$$\varphi_t^*(\omega^n) = (\varphi_t^*\omega)^n = \omega^n$$

$$\int_{T^*M} F \circ \varphi_t = \int_{T^*M} (F \circ \varphi_t)\omega^n = \int_{T^*M} (F \circ \varphi_t)\varphi_t^*(\omega^n)$$

$$= \int_{T^*M} \varphi_t^*(F\omega^n) = \int_{T^*M} F\omega^n = \int_{T^*M} F$$

Poincare recurrence theorem