§ Hamiltonian Mechanics

$v=\sum_i v^i \frac{\partial}{\partial x^i}, \omega=\sum_i p_i dx^i$

$TM\oplus T^*M=\bigcup_{q\in M} T_qM\times T_q^*M$ ，where $v\in T_qM, \omega\in T_q^*M$

M 的局部座標為 $(x^1,x^2,...,x^n)$ 則 $TM\oplus T^*M$ 的局部座標為 $(x^1,...,v^1,...,p_1,...)$

$\pi_1:TM\oplus T^*M\to TM$ ，$\pi_2:TM\oplus T^*M\to TM$

$\pi_1(v,\omega)=v, \pi_2(v,\omega)=\omega$

$\tilde{H}:TM\oplus T^*M\to R$

$\tilde{H}(v,\omega):=\omega(v)-L(v)$ where $L:TM\to R$ is the Lagrangian

$\tilde{H}(x^i,v^i,p_i)=\sum_i p_i v^i - L(x^i,v^i)$

$d\tilde{H}=\sum_i (p_i-\frac{\partial L}{\partial v^i})dv^i+\sum_i v^i dp_i-\sum_i \frac{\partial L}{\partial x^i}dx^i$ ...(1)

因此 把 $\tilde{H}$ 形如 $T_qM\times\{\omega\}$ 的 submanifold 上，$\tilde{H}$ 的 critical point 滿足

$p_i=\frac{\partial L}{\partial v^i}(x^i,v^i)$

假設 $S=\left\{(x^i,v^i,p_i)\in TM\oplus T^*M \,\middle|\, p_i=\frac{\partial L}{\partial v^i}(x^i,v^i)\right\}$

則 $\pi_1|_S:S\to TM$ 是一可微同胚，若 $\pi_2|_S:S\to T^*M$ 一可微同胚 則稱此

Lagrangian L 為 hyper-regular(超正則) 此時 $\pi_2|_S\circ\pi_1|_S^{-1}:TM\to T^*M$ 是一保持 fiber 的可微同胚。

$H=\tilde{H}|_S$ 則 $dH=\sum_i v^i dp_i-\sum_i \frac{\partial L}{\partial x^i}dx^i$ ....(1)式中 $p_i=\frac{\partial L}{\partial v^i}(x^i,v^i)$

又 $dH=\sum_i \frac{\partial H}{\partial x^i}dx^i+\sum_i \frac{\partial H}{\partial p_i}dp_i$ 比較此兩式 得

$$\begin{cases}\frac{\partial H}{\partial x^i}=-\frac{\partial L}{\partial x^i}\\ \frac{\partial H}{\partial p_i}=v^i\end{cases}$$

Proposition

The Euler-Lagrange equations for a hyper-regular Lagrangian L define a flow on M。

This flow is carried by the Legendre transformation to the flow defined on T^*M by the Hamilton equations

$$\begin{cases} \dot{x}^i = \dfrac{\partial H}{\partial p_i} \text{....}(1') \\ \dot{p}_i = -\dfrac{\partial H}{\partial x^i} \text{...}(2') \end{cases}$$

proof

(1)在 S 上 $p_i = \dfrac{\partial L}{\partial v^i}(x^i, v^i)$ (2) $\dot{x}^i = v^i$ (3) E-L equations $\dfrac{d}{dt}(\dfrac{\partial L}{\partial v^i}) = \dfrac{\partial L}{\partial x^i}$

把(1)(2)(3)兜起來即可。

(1')位置隨時間變化 (2')動量隨時間變化

若 $\dfrac{\partial H}{\partial t} = 0$ 則 H 是一守恆量。

P 動量 q:=x 的位置

$L(q, \dot{q}, t)$ 是 Lagrangian 其 Hamiltonian 為

$H(p,q,t) = \dfrac{\partial L}{\partial \dot{q}}\dot{q} - L(q,\dot{q},t) = p\dot{q} - L(q,\dot{q},t)$ ，$\dfrac{\partial L}{\partial \dot{q}}$ 是廣義動量