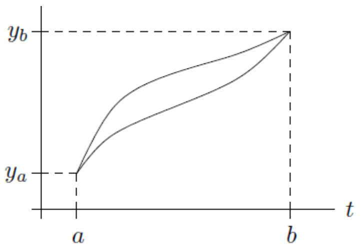


$$I(x)=\int_a^b F(x(t),\dot{x}(t),t)dt : S\to R$$

有極值的必要條件

$$x\in C^1(a,b), x(a)=y_a, x(b)=y_b$$

$$F=F(\alpha,\beta,\gamma)$$

If I has an extremum at $x_0\in S$，then x_0 satisfies the Euler-Lagrange equation

$$\frac{\partial F}{\partial\alpha}(x_0(t),\dot{x}_0(t),t)-\frac{\partial F}{\partial\beta}(x_0(t),\dot{x}_0(t),t)=0$$

The solution of the Euler-Lagrange equation are called critical curves。

If F does not depend on γ，then the Euler-Lagrange equation becomes

$$F(x(t),\dot{x}(t),t)-\dot{x}(t)\frac{\partial F}{\partial\beta}(x(t),\dot{x}(t),t)=C$$ ，where C is a constant。

§ 5.5 Lagrangian Mechanics

Let M is a differentiable manifold， $p,q\in M$ and $a,b\in R$ a<b。
Let us denote C the set of differentiable curves $c:[a,b]\to M$ such that c(a)=p，c(b)=q 。

Definition

$$L:TM\to R \quad ， L=L(x(t),\dot{x}(t))$$

a Lagrangian function on M，and the action determined by L on C is the map

$A:C\to R$ ， $A(c):=\int_a^b L(\dot{c}(t))dt$ 稱為 L 所決定的作用量。

$\gamma:(-\varepsilon,\varepsilon)\times[a,b]\to M$ is a variation of the c such that $\gamma(0,t)=c(t)$

The curve c is said to be the critical point of the action A if $\frac{d}{ds}|_{s=0}A(\gamma(s))=0\Leftrightarrow$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}(x(t),\dot{x}(t))\right)-\frac{\partial L}{\partial x^i}(x(t),\dot{x}(t))=0$$

上式稱為 Euler-Lagrange equations。簡單寫成 $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{u}}\right)-\frac{\partial L}{\partial u}=0$

推論

任何守恆的(conservative)力學系統$(M,<.,.>,-dU)$的運動是由 Lagrangian L:=K-U 所決定的作用量的 critical points。

The fiber derivative is used to convert between the Lagrangian and Hamiltonian forms。If M is the configuration manifold then the Lagrangian L is defined on the tangent bundle TM，and the Hamiltonian is defined on the cotangent bundle T^*M。

$FL: TM \to T^*M$ such that

$$(FL)_v(w) := \frac{d}{dt}\Big|_{t=0} L(v+tw)$$

以及 Hamiltonian function $H: TM \to R$，$H(v) := (FL)_v(v) - L(v)$

在局部座標系中 $H(x,v) = \sum_i v^i \frac{\partial L}{\partial v^i}(x,v) - L(x,v)$

則此 Hamiltonian function H 沿 E-L 方程的解為常數 i.e. $\frac{d}{dt}(H(\dot{c}(t))) = 0$

這件事在 symplectic manifold 中應該有解釋(不同的說法)。

換句話說…

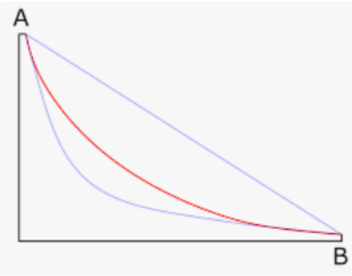
$dH = -i_X \omega$

例

If $(M,<.,.>,-dU)$ is a conservative mechanical system，then its motions are the solutions of the Euler-Lagrange equations for the Lagrangian $L: TM \to R$ given by $L(v) = \frac{1}{2} <v,v> - U(\pi(v))$，and $H(v) = \frac{1}{2} <v,v> + U(\pi(v))$ is the mechanical energy。

後面 諾特定理(Noether)完全不懂

§



最速降線(brachistochrone)問題：擺線(cycloid)

時間$T=\int_{p_1}^{p_2}\frac{ds}{v}$ ，s 是弧長 v 是速度

$$\frac{1}{2}mv^2=mgy\Rightarrow v=\frac{ds}{dt}=\sqrt{2gy}$$

$$ds^2=dx^2+dy^2$$

$$\frac{dt}{ds}=\frac{1}{\sqrt{2gy}}\text{，}dt=\frac{ds}{\sqrt{2gy}}=\frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx}{\sqrt{2gy}}\text{，}T=\int_0^a dt=\int_0^a\frac{\sqrt{1+y'^2}}{\sqrt{2gy}}dx$$

Euler-Lagrange equation 為$\frac{\partial L}{\partial y}-\frac{d}{dx}(\frac{\partial L}{\partial y'})=0$

此式與 x 無關，所以$\frac{\partial L}{\partial x}=0$ 得$L-y'\frac{\partial L}{\partial y'}=c$ ---(*)

把(*)兩邊對 x 微分，得

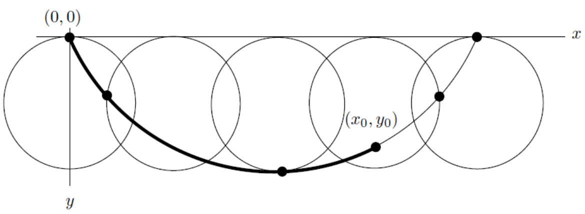
$$\frac{\partial L}{\partial y}y'+\frac{\partial L}{\partial y'}\frac{\partial y'}{\partial x}-\left(\frac{\partial y'}{\partial x}\frac{\partial L}{\partial y'}+y'\frac{d}{dx}(\frac{\partial L}{\partial y'})\right)=0\text{，即}\frac{\partial L}{\partial y}-\frac{d}{dx}(\frac{\partial L}{\partial y'})=0$$

$$L(y,y',x)=\sqrt{\frac{1+y'^2}{y}}$$

$$\sqrt{\frac{1+y'^2}{y}}-\frac{1}{\sqrt{y}}\frac{y'^2}{\sqrt{1+y'^2}}=c\text{，}\frac{1}{\sqrt{y}}\times\frac{1}{\sqrt{1+y'^2}}=c$$

$y(1+y'^2)=\frac{1}{c^2}$ let=c

$$dx=\sqrt{\frac{y}{c-y}}dy\text{，}\Rightarrow\begin{cases}x=\frac{k}{2}(\theta-\sin\theta)\\y=\frac{k}{2}(1-\cos\theta)\end{cases}$$



例 $S=\left\{x\in C^1[0,1]\middle|x(0)=0,x(1)=1\right\}$，

$I(x)=\int_0^1(\frac{d}{dt}x(t)-1)^2dt$ 在 $x_0\in S$ 有最小值，求 x_0

Euler-Lagrange equation is $0-\frac{d}{dt}(2(x_0'(t)-1)=0$ for $\forall t\in[0,1]$

$x_0(t)=At+B$ ，by $x_0(0)=0,x_0(1)=1$

解出 $x_0(t)=t$

習作

1. Find the critical curve for the function $I(x)=\int_1^2 t^3(x'(t))^2dt$

   $x\in C^1[1,2],x(1)=5,x(2)=2$

   $\frac{d}{dt}(2t^3\cdot x'(t))=0$

   $3t^2x'(t)+t^3x''(t)=0$ ，Let $x'=p$ then $tp'+3p=0$

   $\frac{dp}{p}=-\frac{3dt}{t}$ ， $p=x'(t)=ct^{-3}$ ， $x(t)=\frac{ct^{-2}}{2}+d$

   $x_0(t)=4t^{-2}+1$

2. Find critical curves for the function $I(x)=\int_1^2\frac{(x'(t))^3}{t^2}dt$ ，where $x\in C^1[1,2]$ with x(1)=1 and x(2)=7

$x_0(t)=2t^2-1$

3. Find critical curves for the function

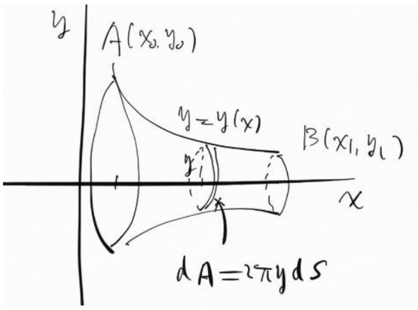
   $I(x)=\int_0^1[2tx(t)-(x'(x))^2+3x'(t)(x(t))^2]dt$ ，where $x\in C^1[0,1]$ with x(0)=0 and x(1)=-1

$\frac{d}{dt}\left(-2x'(t)+3(x(t))^2\right)-\left(2t+6x'(t)x(t)\right)=0$

$x''(t)=-t$

$x(t)=$

4. Find critical curves for the function $I(x)=\int_0^1[2(x(t))^3+3t^2x'(t)]dt$ ，where $x\in C^1[0,1]$ with x(1)=0 and x(1)=2



例 連接 $A(x_0,y_0),B(x_1,y_1)$ 繞 x 軸的旋轉體表面積最小

$A(y)=2\pi\int_{x_0}^{x_1}y(x)\sqrt{1+(y'(x))^2}dx$ ，$dA=2\pi yds, ds^2=dx^2+dy^2$

$\frac{\partial L}{\partial y}-\frac{d}{dx}\frac{\partial L}{\partial y'}=0$ ，$\Downarrow if\ \frac{\partial L}{\partial x}=0$ $L-y'\frac{\partial L}{\partial y'}=c$ ，$L=y\sqrt{1+y'^2}$ then

$y\sqrt{1+y'^2}-y'(\frac{yy'}{\sqrt{1+y'^2}})=c\Rightarrow y=c\sqrt{1+(y')^2}$ 可推出 $y=c\cosh\left(\frac{x+d}{c}\right)$

$\left(\frac{y}{c}\right)^2=1+y'^2, y'=\sqrt{\frac{y^2-c^2}{c^2}}$

$\frac{cdy}{\sqrt{y^2-c^2}}=dx$ ，let y=c coshu，then dy=c sinhudu

$\sqrt{y^2-c^2}=c\sinh u$

$\frac{cdy}{\sqrt{y^2-c^2}}=\frac{c\cdot c\sinh udu}{c\sinh u}=cdu=dx$ ，cu=x+d

$y=c\cosh u=c\cosh\left(\frac{x+d}{c}\right)$

a catenary(旋鏈線)。其中 $\cosh x=\frac{e^x+e^{-x}}{2}$ ，$\sinh x=\frac{e^x-e^{-x}}{2}$