§ simple pendulum 單擺

1. 數學傳播季刊 27 第二期 [單擺]

   http://web.math.sinica.edu.tw/math_media/d272/27205.pdf

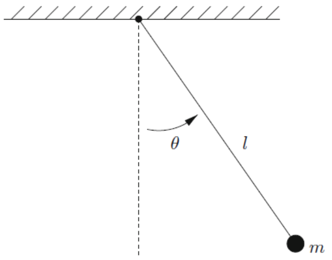
要真正了解單擺運動則有賴於橢圓函數。

2. [An Introduction to Riemannian Geometry] by Leonor Godinho & Jose Natario
3. [Elementary Differential Equations and Boundary Value Problems]

   by William E. Boyce Richard C. DiPrima

上面的支撐點(樞軸點)稱為 pivoting point

位置 $X=(l\cos\theta, l\sin\theta)$



速度 $\dot{X}=(-l\sin\theta, l\cos\theta)\dot{\theta}$

加速度 $\ddot{X}=(-l\cos\theta,-l\sin\theta)\dot{\theta}^2+(-l\sin\theta, l\cos\theta)\ddot{\theta}$

$m\ddot{x}=F=F_a+f$ (前者施力 後者約束力)

$N=\left\{(x,y)\in R^2 \middle| x^2+y^2=l^2\right\}\cong S^1$ holonomic constraint

$\varphi:(-\pi,\pi)\to R^2$, $\varphi(\theta)=(l\sin\theta,-l\cos\theta)$

$$\frac{d}{d\theta}=\frac{dx}{d\theta}\frac{\partial}{\partial x}+\frac{dy}{d\theta}\frac{\partial}{\partial y}=l\cos\theta\frac{\partial}{\partial x}+l\sin\theta\frac{\partial}{\partial y}$$

動能 $K(v\frac{d}{d\theta})=\frac{1}{2}m<vl\cos\theta\frac{\partial}{\partial x}+vl\sin\theta\frac{\partial}{\partial y}, vl\cos\theta\frac{\partial}{\partial x}+vl\sin\theta\frac{\partial}{\partial y}>=\frac{1}{2}ml^2v^2$

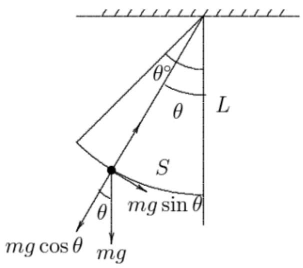
位能 $U(x,y)=mgy=-mgl\cos\theta$

運動方程式

$$\frac{d}{dt}\left(\frac{\partial K}{\partial v}(\theta,\dot{\theta})\right)-\frac{\partial K}{\partial\theta}(\theta,\dot{\theta})=-\frac{\partial U}{\partial\theta}(\theta)\Leftrightarrow\frac{d}{dt}(ml^2\dot{\theta})=-mgl\sin\theta$$

$$\Leftrightarrow\ddot{\theta}=-\frac{g}{l}\sin\theta$$

Lagrangian $L=T-V=\frac{1}{2}ml^2\dot{\theta}^2+mgl\cos\theta$



運動方程式$\frac{d^2\theta}{dt^2}+\frac{g}{l}\sin\theta=0$的解釋

(一) $F=-mg\sin\theta=ma$，所以$a=-g\sin\theta$

$s=l\theta$，$v=\frac{ds}{dt}=l\frac{d\theta}{dt}$，$a=\frac{d^2s}{dt^2}=l\frac{d^2\theta}{dt^2}$

所以$l\frac{d^2\theta}{dt^2}=-g\sin\theta$，i.e. $\frac{d^2\theta}{dt^2}+\frac{g}{l}\sin\theta=0$

(二) $\Delta U=mgh$，$\Delta K=\frac{1}{2}mv^2$，$\frac{1}{2}mv^2=mgh$

$v=\sqrt{2gh}=l\frac{d\theta}{dt}$，$\therefore\frac{d\theta}{dt}=\frac{\sqrt{2gh}}{l}=\sqrt{\frac{2g}{l}(\cos\theta-\cos\theta_0)}$

$(h=l(\cos\theta-\cos\theta_0))$

$$\frac{d^2\theta}{dt^2}=\frac{1}{2}\frac{-\frac{2g}{l}\sin\theta}{\sqrt{\frac{2g}{l}(\cos\theta-\cos\theta_0)}}\frac{d\theta}{dt}=-\frac{g}{l}\sin\theta$$

θ很小時，$\sin\theta\approx\theta$，變成簡諧運動。

§ 解微分方程 $\frac{d^2\theta}{dt^2}=-k\sin\theta, k=\frac{g}{l}$

[Elementary Differential Equations and Boundary Value Problems]

by William E. Boyce & Richard C. DiPrima

p.497 p.503 almost linear system p.508

1. The equation of motion of an undamped pendulum is $\frac{d^2\theta}{dt^2}+\omega^2\sin\theta=0$，

   where $\omega^2=\frac{g}{l}$。Let $x=\theta, y=\frac{d\theta}{dt}$ to obtain the system of equations

$$\begin{cases}\frac{dx}{dt}=y\\ \frac{dy}{dt}=-\omega^2\sin x\end{cases}$$

   (a) Show that the critical points are $(\pm n\pi,0), n=0,1,2,...$，and that the system is almost linear in the neighborhood of each critical point。

   (b) Show that the critical point (0,0) is a (stable)center of the corresponding linear system。The situation is similar to the critical points $(\pm 2n\pi,0)$，n=1,2,3,...What is the physical interpretation of these critical points？

(c) Show that the critical point $(\pi, 0)$ is an(unstable) saddle point of the corresponding linear system。What conclusion can you draw about the nonlinear system？The situation is similar to the critical points $(\pm(2n-1)\pi, 0), n = 1, 2, 3, \ldots$ What is the physical interpretation of these critical points？

(d) Choose a value for ω^2 and plot a few trajectories of the nonlinear system in the neighborhood of the origin。Can you now draw any further conclusion about he nature of the critical point at (0,0) for the nonlinear system？

(e) Using the value of ω^2 from part(d)，draw a phase portrait for the pendulum。

20. (a) By solving the equation for dy/dx, show that the equation of the trajectories of the undamped pendulum of Problem 19 can be written as

$$\tfrac{1}{2}y^2 + \omega^2(1 - \cos x) = c, \qquad \text{(i)}$$

where c is a constant of integration.

(b) Multiply Eq. (i) by mL^2. Then express the result in terms of θ to obtain

$$\frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 + mgL(1 - \cos\theta) = E, \qquad \text{(ii)}$$

where $E = mL^2c$.

2.

(c) Show that the first term in Eq. (ii) is the kinetic energy of the pendulum and that the second term is the potential energy due to gravity. Thus the total energy E of the pendulum is constant along any trajectory; its value is determined by the initial conditions.

21. The motion of a certain undamped pendulum is described by the equations

$$dx/dt = y, \qquad dy/dt = -4\sin x.$$

If the pendulum is set in motion with an angular displacement A and no initial velocity, then the initial conditions are $x(0) = A, y(0) = 0$.

(a) Let $A = 0.25$ and plot x versus t. From the graph, estimate the amplitude R and period T of the resulting motion of the pendulum.

(b) Repeat part (a) for $A = 0.5, 1.0, 1.5$, and 2.0.

(c) How do the amplitude and period of the pendulum's motion depend on the initial position A? Draw a graph to show each of these relationships. Can you say anything about the limiting value of the period as $A \to 0$?

(d) Let $A = 4$ and plot x versus t. Explain why this graph differs from those in parts (a) and (b). For what value of A does the transition take place?

3.

22. Consider again the pendulum equations (see Problem 21)

$$dx/dt = y, \qquad dy/dt = -4\sin x.$$

If the pendulum is set in motion from its downward equilibrium position with angular velocity v, then the initial conditions are $x(0) = 0, y(0) = v$.

(a) Plot x versus t for $v = 2$ and also for $v = 5$. Explain the differing motions of the pendulum that these two graphs represent.

(b) There is a critical value of v, which we denote by v_c, such that one type of motion
4. occurs for $v < v_c$ and the other for $v > v_c$. Estimate the value of v_c.

23. This problem extends Problem 22 to a damped pendulum. The equations of motion are

$$dx/dt = y, \qquad dy/dt = -4\sin x - \gamma y,$$

where γ is the damping coefficient, with the initial conditions $x(0) = 0, y(0) = v$.

(a) For $\gamma = 1/4$ plot x versus t for $v = 2$ and for $v = 5$. Explain these plots in terms of the motions of the pendulum that they represent. Also explain how they relate to the corresponding graphs in Problem 22(a).

(b) Estimate the critical value v_c of the initial velocity where the transition from one type of motion to the other occurs.

(c) Repeat part (b) for other values of γ and determine how v_c depends on γ.

27. In this problem we derive a formula for the natural period of an undamped nonlinear pendulum [$c = 0$ in Eq. (10) of Section 9.2]. Suppose that the bob is pulled through a positive angle α and then released with zero velocity.

(a) We usually think of θ and $d\theta/dt$ as functions of t. However, if we reverse the roles of t and θ, we can regard t as a function of θ and, consequently, can also think of $d\theta/dt$ as a function of θ. Then derive the following sequence of equations:

$$\frac{1}{2}mL^2\frac{d}{d\theta}\left[\left(\frac{d\theta}{dt}\right)^2\right] = -mgL\sin\theta,$$

$$\frac{1}{2}m\left(L\frac{d\theta}{dt}\right)^2 = mgL(\cos\theta - \cos\alpha),$$

$$dt = -\sqrt{\frac{L}{2g}}\frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}.$$

Why was the negative square root chosen in the last equation?

(b) If T is the natural period of oscillation, derive the formula

$$\frac{T}{4} = -\sqrt{\frac{L}{2g}} \int_{\alpha}^{0} \frac{d\theta}{\sqrt{\cos\theta - \cos\alpha}}.$$

(c) By using the identities $\cos\theta = 1 - 2\sin^2(\theta/2)$ and $\cos\alpha = 1 - 2\sin^2(\alpha/2)$, followed by the change of variable $\sin(\theta/2) = k\sin\phi$ with $k = \sin(\alpha/2)$, show that

$$T = 4\sqrt{\frac{L}{g}} \int_{0}^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}.$$

The integral is called the **elliptic integral** of the first kind. Note that the period depends on the ratio L/g and also on the initial displacement α through $k = \sin(\alpha/2)$.

(d) By evaluating the integral in the expression for T, obtain values for T that you can compare with the graphical estimates you obtained in Problem 21.

28. A generalization of the damped pendulum equation discussed in the text, or a damped spring–mass system, is the Liénard[2] equation

$$\frac{d^2x}{dt^2} + c(x)\frac{dx}{dt} + g(x) = 0.$$

If $c(x)$ is a constant and $g(x) = kx$, then this equation has the form of the linear pendulum equation [replace $\sin\theta$ with θ in Eq. (12) of Section 9.2]; otherwise, the damping force $c(x)\,dx/dt$ and the restoring force $g(x)$ are nonlinear. Assume that c is continuously differentiable, g is twice continuously differentiable, and $g(0) = 0$.

(a) Write the Liénard equation as a system of two first order equations by introducing the variable $y = dx/dt$.

(b) Show that $(0, 0)$ is a critical point and that the system is almost linear in the neighborhood of $(0, 0)$.

(c) Show that if $c(0) > 0$ and $g'(0) > 0$, then the critical point is asymptotically stable, and that if $c(0) < 0$ or $g'(0) < 0$, then the critical point is unstable.
Hint: Use Taylor series to approximate c and g in the neighborhood of $x = 0$.

参考 Lyapunov's method