例.Ice skate

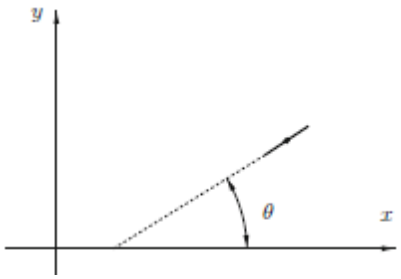
單輪車沿自己的方向移動或者以它的中心轉動，

位相空間(Configuration space)是 $R^2 \times S^1$

$X = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$ ， $Y = \frac{\partial}{\partial \theta}$ ，平面場 $\Sigma = \{X, Y\}$

The kernel of the 1-form $\omega = -\sin\theta dx + \cos\theta dy$ (constraint 約束)

(ω 與 X 是對偶關係 所以看起來像直線的斜率與方向向量的關係)



$d\omega = -\cos\theta d\theta \wedge dx - \sin\theta d\theta \wedge dy$

$d\omega \wedge \omega = -d\theta \wedge dx \wedge dy \neq 0$ 因此 ω 是不可積約束（non-holonomic constraint）。

換句話說 Σ 是一不可積平面場(non-integrable distribution)

(a) Show that the ice skate can access all points in the configuration space: given two points $p, q \in \mathbb{R}^2 \times S^1$ there exists a piecewise smooth curve $c : [0, 1] \to \mathbb{R}^2 \times S^1$ compatible with Σ such that $c(0) = p$ and $c(1) = q$. Why does this show that Σ is non-integrable?

(b) Assuming that the kinetic energy of the skate is

$$K = \frac{M}{2}\left((v^x)^2 + (v^y)^2\right) + \frac{I}{2}(v^\theta)^2$$

and that the reaction force is perfect, show that the skate moves with constant speed along straight lines or circles. What is the physical interpretation of the reaction force?

(c) Determine the motion of the skate moving on an inclined plane, i.e. subject to a potential energy $U = Mg \sin\alpha\, x$.