Dirac Operators and Spectral Geometry

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Appendix A

Exercises

A.1 Examples of Dirac operators

A.1.1 The circle

Let $M := \mathbb{S}^1$, regarded as $\mathbb{S}^1 \simeq \mathbb{R}/\mathbb{Z}$; that is to say, we parametrize the circle by the half-open interval $[0, 1)$ rather than $[0, 2\pi)$, say. Then $\mathcal{A} = C^\infty(\mathbb{S}^1)$ can be identified with periodic smooth functions on $\mathbb{R}$ with period 1:

$$\mathcal{A} \simeq \{\, f \in C^\infty(\mathbb{R}) : f(t+1) \equiv f(t) \,\}.$$

Since $\mathrm{Cl}(\mathbb{R}) = \mathbb{C}1 \oplus \mathbb{C}e_1$ as a $\mathbb{Z}_2$-graded algebra, we see that $\mathcal{B} = \mathcal{A}$ in this case; and since $n = 1$, $m = 0$ and $2^m = 1$, there is a "trivial" spin structure given by $\mathcal{S} := \mathcal{A}$ itself. The charge conjugation is just $C = K$, where K means *complex conjugation* of functions. With the flat metric on the circle, the Dirac operator is just

$$D\!\!\!/ := -i \, \frac{d}{dt}.$$

Exercise A.1. *Show that its spectrum is*

$$\mathrm{sp}(D\!\!\!/) = 2\pi\mathbb{Z} = \{\, 2\pi k : k \in \mathbb{Z} \,\},$$

by first checking that the eigenfunctions $\psi_k(t) := e^{2\pi i k t}$ form an orthonormal basis for the Hilbert-space completion $\mathcal{H}$ of $\mathcal{S}$ – using Fourier series theory.

The point is that the closed span of these eigenvectors is all of $\mathcal{H}$, so that $\mathrm{sp}(D\!\!\!/)$ contains no more than the corresponding eigenvalues.

Next, consider

$$\mathcal{S}' := \{\, \phi \in C^\infty(\mathbb{R}) : \phi(t+1) \equiv -\phi(t) \,\},$$

which can be thought of as the space of smooth functions on the interval $[0, 1]$ "with antiperiodic boundary conditions".

Exercise A.2. *Explain in detail how $\mathcal{S}'$ can be regarded as a $\mathcal{B}$-$\mathcal{A}$-bimodule, and how $C = K$ acts on it as a charge-conjugation operator. Taking $D\!\!\!/ := -i\, d/dt$ again, but now as an operator with domain $\mathcal{S}'$ on the Hilbert-space completion of $\mathcal{S}'$, show that its spectrum is now*

$$\mathrm{sp}(D\!\!\!/) = 2\pi(\mathbb{Z} + \tfrac{1}{2}) = \{\, \pi(2k+1) : k \in \mathbb{Z} \,\},$$

by checking that $\phi_k(t) := e^{\pi i (2k+1) t}$ are a complete set of eigenfunctions.

The circle $\mathbb{S}^1$ thus carries two inequivalent spin structures: their inequivalence is most clearly manifest in the different spectra of the Dirac operators. Notice that $0 \in \operatorname{sp}(D\!\!\!/)$ for the "untwisted" spin structure where $\mathcal{S} = \mathcal{A}$, while $0 \notin \operatorname{sp}(D\!\!\!/)$ for the "twisted" spin structure whose spinor module is $\mathcal{S}'$. There are no more spin structures to be found, since $H^1(\mathbb{S}^1, \mathbb{Z}_2) = \mathbb{Z}_2$.

A.1.2 The (flat) torus

On the 2-torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$, we use the Riemannian metric coming from the usual flat metric on $\mathbb{R}^2$. Thus, if we regard $\mathcal{A} = C^\infty(\mathbb{T}^2)$ as the smooth periodic functions on $\mathbb{R}^2$ with $f(t^1, t^2) \equiv f(t^1+1, t^2) \equiv f(t^1, t^2+1)$, then (t^1, t^2) define local coordinates on $\mathbb{T}^2$, with respect to which all Christoffel symbols are zero, namely $\Gamma^k_{ij} = 0$, and thus $\nabla = d$ represents the Levi-Civita connection on 1-forms.

In this case, $n = 2$, $m = 1$ and $2^m = 2$, so we use "two-component" spinors; that is, the spinor bundles $S \to \mathbb{T}^2$ are of rank two. There is the "untwisted" one, where S is the trivial rank-two $\mathbb{C}$-vector bundle, and $\mathcal{S} \simeq \mathcal{A}^2$. The Clifford algebra in this case is just $\mathcal{B} = M_2(\mathcal{A})$. Using the standard Pauli matrices:

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we can write the charge conjugation operator as

$$C = -i\,\sigma^2\,K$$

where K again denotes (componentwise) complex conjugation.

Exercise A.3. *Find three more spinor structures on $\mathbb{T}^2$, exhibiting each spinor module as a $\mathcal{B}$-$\mathcal{A}$-bimodule, with the appropriate action of C. (Use $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$.)*

Exercise A.4. *Check that*

$$D\!\!\!/ = -i\,(\sigma^1\,\partial_1 + \sigma^2\,\partial_2) = \begin{pmatrix} 0 & -\partial_2 - i\partial_1 \\ \partial_2 - i\partial_1 & 0 \end{pmatrix}$$

where $\partial_1 = \partial/\partial t^1$ and $\partial_2 = \partial/\partial t^2$, is indeed the Dirac operator on the untwisted spinor module $\mathcal{S} = \mathcal{A}^2$. Compute $\operatorname{sp}(D\!\!\!/^2)$ *by finding a complete set of eigenvectors. Then show that*

$$\operatorname{sp}(D\!\!\!/) = \{\, \pm 2\pi\sqrt{r_1^2 + r_2^2} : (r_1, r_2) \in \mathbb{Z}\,\}$$

by finding the eigenspinors for each of these eigenvalues. ¿What can be said of the multiplicities *of these eigenvalues? and what is the dimension of* $\ker D\!\!\!/$*?*

Notice that σ^3 does not appear in the formula for $D\!\!\!/$; its role here is to give the $\mathbb{Z}_2$-grading operator: $c(\gamma) = \sigma^3$ —regarded as a constant function with values in $M_2(\mathbb{C})$— in view of the relation $\sigma^3 = -i\,\sigma^1\sigma^2$ among Pauli matrices.

On the 3-torus $\mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3$, where now $n = 3$, $m = 1$ and again $2^m = 2$, we get two-component spinors. Again we may use a flat metric and an untwisted spin structure with $\mathcal{S} = \mathcal{A}^2$. The charge conjugation is still $C = -i\,\sigma^2\,K$ on $\mathcal{S}$, so that $C^2 = -1$ also in this 3-dimensional case. The Dirac operator is now

$$D\!\!\!/ = -i\,(\sigma^1\,\partial_1 + \sigma^2\,\partial_2 + \sigma^3\,\partial_3) = \begin{pmatrix} -i\,\partial_3 & -\partial_2 - i\partial_1 \\ \partial_2 - i\partial_1 & i\,\partial_3 \end{pmatrix}.$$

Exercise A.5. *Compute* $\operatorname{sp}(D\!\!\!/^2)$ *and* $\operatorname{sp}(D\!\!\!/)$ *for this Dirac operator on $\mathbb{T}^3$.*

A.1.3 The Hodge–Dirac operator on $\mathbb{S}^2$

If M is a compact, oriented Riemannian manifold that has no spin^c structures, ¿can one define Dirac-like operators on an $\mathcal{B}$-$\mathcal{A}$-bimodule $\mathcal{E}$ that is not pointwise irreducible under the action of $\mathcal{B}$? It turns out that one can do so, if $\mathcal{E}$ carries a "Clifford connection", that is, a connection $\nabla^{\mathcal{E}}$ such that

$$\nabla^{\mathcal{E}}(c(\alpha)s) = c(\nabla\alpha)\, s + c(\alpha)\, \nabla^{\mathcal{E}} s,$$

for $\alpha \in \mathcal{A}^1(M)$, $s \in \mathcal{E}$, and which is Hermitian with respect to a suitable $\mathcal{A}$-valued sesquilinear pairing on $\mathcal{E}$. For instance, we may take $\mathcal{E} = \mathcal{A}^\bullet(M)$, the full algebra of differential forms on M, which we know to be a left $\mathcal{B}$-module under the action generated by $c(\alpha) = \varepsilon(\alpha) + \iota(\alpha^\sharp)$. The Clifford connection is just the Levi-Civita connection on all forms, obtaining by extending the one on $\mathcal{A}^1(M)$ with the Leibniz rule (and setting $\nabla f := df$ on functions). The pairing $(\alpha \mid \beta) := g(\bar\alpha, \beta)$ extends to a pairing on $\mathcal{A}^\bullet(M)$; by integrating the result over M with respect to the volume form ν_g, we get a scalar product on forms, and we can then complete $\mathcal{A}^\bullet(M)$ to a Hilbert space.

If $\{E_1, \dots, E_n\}$ and $\{\theta^1, \dots, \theta^n\}$ are local orthonormal sections for $\mathcal{X}(M)$ and $\mathcal{A}^1(M)$ respectively, compatible with the given orientation, so that $c(\theta^j) = \varepsilon(\theta^j) + \iota(E_j)$ locally, then

$$\star := c(\gamma) = (-i)^m\, c(\theta^1)\, c(\theta^2) \dots c(\theta^n)$$

is globally well-defined as an $\mathcal{A}$-*linear operator* taking $\mathcal{A}^\bullet(M)$ onto itself, such that $\star^2 = 1$. This is the *Hodge star* operator, and it exchanges forms of high and low degree.

Exercise A.6. *If $\{1, \dots, n\} = \{i_1, \dots, i_k\} \uplus \{j_1, \dots, j_{n-k}\}$, show that locally,*

$$\star(\theta^{i_1} \wedge \dots \wedge \theta^{i_k}) = \pm i^m\, \theta^{j_1} \wedge \dots \wedge \theta^{j_{n-k}},$$

where the sign depends on $i_1, \dots, i_k$. Conclude that $\star$ maps $\mathcal{A}^k(M)$ onto $\mathcal{A}^{n-k}(M)$, for each $k = 0, 1, \dots, n$.

(Actually, our sign conventions differ from the usual ones in differential geometry books, that do not include the factor $(-i)^m$. With the standard conventions, $\star^2 = \pm 1$ on each $\mathcal{A}^k(M)$, with a sign depending on the degree k.)

The *codifferential* δ on $\mathcal{A}^\bullet(M)$ is defined by

$$\delta := -\star d \star.$$

This operation *lowers* the form degree by 1. The **Hodge–Dirac** operator is defined to be $-i(d+\delta)$ on $\mathcal{A}^\bullet(M)$. One can show that, on the Hilbert-space completion, the operators d and $-\delta$ are adjoint to one another, so that $-i(d+\delta)$ extends to a selfadjoint operator. (With the more usual sign conventions, d and $+\delta$ are adjoint, so that the Hodge–Dirac operator is written simply $d + \delta$.)

Now we take $M = \mathbb{S}^2$, the 2-sphere of radius 1. The round (i.e., rotation-invariant) metric on $\mathbb{S}^2$ is written $g = d\theta^2 + \sin^2\theta\, d\phi^2$ in the usual spherical coordinates, which means that $\{d\theta, \sin\theta\, d\phi\}$ is a local orthonormal basis of 1-forms on $\mathbb{S}^2$. The area form is $\nu = \sin\theta\, d\theta \wedge d\phi$. The Hodge star is specified by defining it on 1 and on $d\theta$:

$$\star(1) := -i\,\nu, \qquad \star(d\theta) := i \sin\theta\, d\phi.$$

To find the eigenforms of the Hodge–Dirac operator, it is convenient to use another set of coordinates, obtained form the Cartesian relation $(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$ by setting $\zeta := x^1 + ix^2 = e^{i\phi}\cos\theta$, along with $x^3 = \cos\theta$; the pair (ζ, x^3) can serve as coordinates for $\mathbb{S}^2$, subject to the relation $\zeta\bar\zeta + (x^3)^2 = 1$. (The extra variable $\bar\zeta$ gives a third coordinate, extending $\mathbb{S}^2$ to $\mathbb{R}^3$.)

Exercise A.7. *Check that in the* (ζ, x^3) *coordinates, the Hodge star is given by*

$$\star(\zeta) = -i\, d\zeta \wedge dx^3, \qquad \star(d\zeta) = x^3\, d\zeta - \zeta\, dx^3.$$

Exercise A.8. *Consider the (complex) vectorfields on* $\mathbb{R}^3$ *given by*

$$L_+ := 2ix^3 \frac{\partial}{\partial \bar\zeta} + i\zeta \frac{\partial}{\partial x^3}, \quad L_- := 2ix^3 \frac{\partial}{\partial \zeta} - i\bar\zeta \frac{\partial}{\partial x^3}, \quad L_3 := i\bar\zeta \frac{\partial}{\partial \bar\zeta} - i\zeta \frac{\partial}{\partial \zeta}.$$

Verify the commutation relations $[L_+, L_-] = -2i\, L_3,\ \ [L_3, L_-] = i\, L_-$ *and* $[L_3, L_+] = -i\, L_+$.

These commutation relations show that if $L_\pm =: L_1 \pm i\, L_2$, then L_1, L_2, L_3 generate a representation of the Lie algebra of the rotation group SO(3). One obtains representation spaces of $SO(3)$ by finding functions f_0 ("highest weight vectors") such that $L_3 f_0$ is a multiple of f_0, $L_+ f_0 = 0$, and $\{\, (L_-)^r f_0 : r \in \mathbb{N}\,\}$ spans a space of finite dimension. To get spaces of differential forms with these properties, one extends each vector field L_j to an operator on $\mathcal{A}^\bullet(\mathbb{S}^2)$, namely its *Lie derivative* $\mathcal{L}_j$, just by requiring that $\mathcal{L}_j\, d = d\, \mathcal{L}_j$. Since the Hodge star operator is unchanged by applying a rotation to an orthonormal basis of 1-forms, one can also show that $\mathcal{L}_j \star = \star \mathcal{L}_j$, so that the Hodge–Dirac operator $-i(d+\delta)$ commutes with each $\mathcal{L}_j$. This gives a method of finding subspaces of joint eigenforms for each eigenvalue of the Hodge–Dirac operator.

We introduce the following families of forms:

$$\begin{aligned} \phi_l^+ &:= i\zeta^l (1 - i\nu), \quad l = 0,1,2,3,\dots; & \psi_l^+ &:= \zeta^{l-1}(d\zeta + \star(d\zeta)), \quad l = 1,2,3,\dots; \\ \phi_l^- &:= i\zeta^l (1 + i\nu), \quad l = 0,1,2,3,\dots; & \psi_l^- &:= \zeta^{l-1}(d\zeta - \star(d\zeta)), \quad l = 1,2,3,\dots. \end{aligned}$$

Clearly, $\star(\phi_l^\pm) = \pm\phi_l^\pm$ and $\star(\psi_l^\pm) = \pm\psi_l^\pm$. Thus ϕ_l^+ and ψ_l^+ are even, while ϕ_l^- and ψ_l^- are odd, with respecting to the $\mathbb{Z}_2$-grading on forms given by $\mathcal{A}^\bullet(\mathbb{S}^2) = \mathcal{A}^+(\mathbb{S}^2) \oplus \mathcal{A}^-(\mathbb{S}^2)$, where $\mathcal{A}^\pm(\mathbb{S}^2) := \frac{1}{2}(1 \pm \star)\, \mathcal{A}^\bullet(\mathbb{S}^2)$.

Exercise A.9. *Show that*

$$\begin{aligned} -i(d+\delta)\phi_l^\pm &= l\psi^\mp, \quad \text{for } l = 0,1,2,\dots \\ -i(d+\delta)\psi_l^\pm &= (l+1)\phi^\mp, \quad \text{for } l = 1,2,3,\dots \end{aligned}$$

and conclude that each of ϕ_l^+, ϕ_l^-, ψ_l^+ *and* ψ_l^- *is an eigenvector for* $(-i(d+\delta))^2 = -(d\delta + \delta d)$ *with eigenvalue* $l(l+1)$. *Find corresponding eigenspinors for* $-i(d+\delta)$ *with eigenvalues* $\pm\sqrt{l(l+1)}$.

Exercise A.10. *Show that* $L_3(\zeta^l) = -il\, \zeta^l$, $L_+\zeta^l = 0$, *and that* $(L_-)^k(\zeta^l)$ *is a linear combination of terms* $(x^3)^{k-2r}\bar\zeta^r \zeta^{l-k+r}$ *that does not vanish for* $k = 0,1,\dots,2l$, *and that* $(L_-)^{2l+1}(\zeta^l) = 0$. *Check that* $L_+(L_-)^k(\zeta^l)$ *is a multiple of* $(L_-)^{k-1}(\zeta^l)$, *for* $k = 1,\dots,2l$.

Exercise A.11. *Show that*

$$\mathcal{L}_3\phi_l^\pm = -il\, \phi_l^\pm, \quad \mathcal{L}_+\phi_l^\pm = 0; \qquad \mathcal{L}_3\psi_l^\pm = -il\, \psi_l^\pm, \quad \mathcal{L}_+\psi_l^\pm = 0;$$

for each possible value of l. *Conclude that the forms* $\mathcal{L}_-^k(\phi_l^\pm)$ *and* $\mathcal{L}_-^k(\psi_l^\pm)$ *vanish if and only if* $k \geq 2l+1$. *¿What can now be said about the multiplicities of the eigenvalues of* $-i(d+\delta)$*?*

With some more works, it can be shown that all these eigenforms span a dense subspace of the Hilbert-space completion of $\mathcal{A}^\bullet(\mathbb{S}^2)$, so that these eigenvalues in fact give the full spectrum of the Hodge–Dirac operator.

A.2 The Dirac operator on the sphere $\mathbb{S}^2$

A.2.1 The spinor bundle S on $\mathbb{S}^2$

Consider the 2-dimensional sphere $\mathbb{S}^2$, with its usual orientation, $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\} \simeq \mathbb{C}P^1$. The usual spherical coordinates on $\mathbb{S}^2$ are

$$p = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \in \mathbb{S}^2.$$

The poles are $N = (0,0,1)$ and $S = (0,0,-1)$. Let $U_N = \mathbb{S}^2 \setminus \{N\}$, $U_S = \mathbb{S}^2 \setminus \{S\}$ be the two charts on $\mathbb{S}^2$. Consider the stereographic projections $p \mapsto z : U_N \to \mathbb{C}$, $p \mapsto \zeta : U_S \to \mathbb{C}$ given by

$$z := e^{-i\phi} \cot\frac{\theta}{2}, \qquad \zeta := e^{+i\phi} \tan\frac{\theta}{2},$$

so that $\zeta = 1/z$ on $U_N \cap U_S$. Write

$$q := 1 + z\bar{z} = \frac{2}{1-\cos\theta}, \quad \text{and} \quad q' := 1 + \zeta\bar{\zeta} = \frac{q}{z\bar{z}}.$$

The sphere $\mathbb{S}^2$ has only the "trivial" spin structure $\mathcal{S} = \Gamma(\mathbb{S}^2, S)$, where $S \to \mathbb{S}^2$ has rank two. Now $S = S^+ \oplus S^-$, where $S^\pm \to \mathbb{S}^2$ are complex *line bundles*, and these may be (and are) nontrivial. We argue that $S^+ \to \mathbb{S}^2$ is the "tautological" line bundle coming from $\mathbb{S}^2 \simeq \mathbb{C}P^1$. We know already that

$$\mathcal{S}^\sharp \simeq \mathcal{S} \iff S^* \simeq S \impliedby (S^+)^* \simeq S^-$$

and the converse $S^* \simeq S \implies (S^+)^* \simeq S^-$ will hold provided we can show that $S^\pm \to \mathbb{S}^2$ are nontrivial line bundles. (Otherwise, S^+ and S^- would each be selfdual, but we know that the only selfdual line bundle on $\mathbb{S}^2$ is the trivial one, since $H^2(\mathbb{S}^2, \mathbb{Z}) \simeq \mathbb{Z}$.)

Consider now the (tautological) line bundle $L \to \mathbb{S}^2$, where

$$L_z := \{\, (\lambda z_0, \lambda z_1) \in \mathbb{C}^2 : \lambda \in \ bC \,\}, \quad \text{if } z = \frac{z_1}{z_0}, \qquad L_\infty := \{\, (0,\lambda) \in \mathbb{C}^2 : \lambda \in \mathbb{C} \,\}.$$

In other words, L_z is the complex line through the point $(1, z)$, for $z \in \mathbb{C}$. A particular *local* section of L, defined over U_N, is $\sigma_N(z) := (q^{-\frac12}, zq^{-\frac12})$, which is normalized so that $(\sigma_N \mid \sigma_N) = q^{-1}(1 + \bar{z}z) = 1$ on U_N: this hermitian pairing on $\Gamma(\mathbb{S}^2, L)$ comes from the standard scalar product on $\mathbb{C}^2$ —each L_z is a line in $\mathbb{C}^2$.

Let also $\sigma_S(\zeta) := (\zeta q'^{-\frac12}, q'^{-\frac12})$, normalized so that $(\sigma_S \mid \sigma_S) = 1$ on U_S. Now if $z \neq 0$, then

$$\sigma_S(z^{-1}) = \left(\frac{1}{z\sqrt{q'}}, \frac{1}{\sqrt{q'}} \right) = (\bar{z}/z)^{1/2} \left(\frac{1}{\sqrt{q}}, \frac{z}{\sqrt{q}} \right) = (\bar{z}/z)^{1/2}\, \sigma_N(z).$$

To avoid ambiguity, we state that $(\bar{z}/z)^{1/2}$ means $e^{-i\phi}$, and also $(z/\bar{z})^{1/2}$ will mean $e^{+i\phi}$.

A smooth section of L is given by two functions $\psi_N^+(z,\bar{z})$ and $\psi_S^+(\zeta,\bar{\zeta})$ satisfying the relation $\psi_N^+(z,\bar{z})\sigma_N(z) = \psi_S^+(\zeta,\bar{\zeta})\sigma_S(\zeta)$ on $U_N \cap U_S$. Thus we argue that

$$\psi_N^+(z,\bar{z}) = (\bar{z}/z)^{1/2} \psi_S^+(z^{-1}, \bar{z}^{-1}) \quad \text{for } z \neq 0,$$

and ψ_N^+, ψ_S^+ are regular at $z = 0$ or $\zeta = 0$ respectively. Likewise, a pair of smooth functions ψ_N^-, ψ_S^- on $\mathbb{C}$ is a section of the dual line bundle $L^* \to \mathbb{S}^2$ if and only if

$$\psi_N^-(z,\bar{z}) = (z/\bar{z})^{1/2} \psi_S^-(z^{-1}, \bar{z}^{-1}) \quad \text{for } z \neq 0.$$

We claim now that we can identify $S^+ \simeq L$ and $S^- \simeq L^* = L^{-1}$ —here the notation L^{-1} means that $[L^{-1}]$ is the inverse of $[L]$ in the Picard group $H^2(\mathbb{S}^2, \mathbb{Z})$ that classifies $\mathbb{C}$-line bundles— so that a *spinor* in $\mathbb{S} = \Gamma(\mathbb{S}^2, S)$ is given precisely by two pairs of smooth functions

$$\begin{pmatrix} \psi_N^+(z, \bar{z}) \\ \psi_N^-(z, \bar{z}) \end{pmatrix} \quad \text{on } U_N, \qquad \begin{pmatrix} \psi_S^+(\zeta, \bar{\zeta}) \\ \psi_S^-(\zeta, \bar{\zeta}) \end{pmatrix} \quad \text{on } U_S,$$

satisfying the above transformation rules. (The nontrivial thing is that the spinor components must both be regular at the south pole $z = 0$ and the north pole $\zeta = 0$, respectively.)

Since $\mathcal{S} \otimes_{\mathbb{A}} \mathcal{S}^* \simeq \mathrm{End}_{\mathcal{A}}(\mathcal{S}) \simeq \mathcal{B} \simeq \mathcal{A}^\bullet(\mathbb{S}^2)$ as $\mathcal{A}$-module isomorphisms (we know that $\mathcal{B} \simeq \mathcal{A}^\bullet(\mathbb{S}^2)$ as sections of *vector* bundles), it is enough to show that, as vector bundles,

$$\mathcal{A}^\bullet(\mathbb{S}^2) \simeq L^0 \oplus L^2 \oplus L^{-2} \oplus L^0,$$

where $L^2 = L \otimes L$, $L^{-2} = L^* \otimes L^*$, and $L^0 = \mathbb{S}^2 \times \mathbb{C}$ is the trivial line bundle. It is clear that $\mathcal{A}^0(\mathbb{S}^2) = C^\infty(\mathbb{S}^2) = \mathcal{A} = \Gamma(\mathbb{S}^2, L^0)$; and furthermore, $\mathcal{A}^2(\mathbb{S}^2) \simeq \mathcal{A} = \Gamma(\mathbb{S}^2, L^0)$ since $\Lambda^2 T^*\mathbb{S}^2$ has a nonvanishing global section, namely the volume form $\nu = \sin\theta d\theta \wedge d\phi$.

With respect to the "round" metric on $\mathbb{S}^2$, namely,

$$g := d\theta^2 + \sin^2\theta\, d\phi^2 = \frac{4}{q^2}\,(dx^1 \otimes dx^1 + dx^2 \otimes dx^2),$$

the pairs of 1-forms $\left\{\dfrac{dz}{q}, \dfrac{d\bar{z}}{q}\right\}$ and $\left\{-\dfrac{d\zeta}{q'}, -\dfrac{d\bar{\zeta}}{q'}\right\}$ are local bases for $\mathcal{A}^1(\mathbb{S}^2)$, over U_N and U_S respectively.

Exercise A.12. *Write, for $\alpha \in \mathcal{A}^1(\mathbb{S}^2)$,*

$$\begin{aligned} \alpha &=: f_N(z, \bar{z})\,\frac{dz}{q} + g_N(z, \bar{z})\,\frac{d\bar{z}}{q} \quad \text{on } U_N, \\ &=: -f_S(\zeta, \bar{\zeta})\,\frac{d\zeta}{q'} - g_S(\zeta, \bar{\zeta})\,\frac{d\bar{\zeta}}{q'} \quad \text{on } U_S. \end{aligned}$$

Show that

$$\begin{aligned} f_N(z, \bar{z}) &= (\bar{z}/z)\, f_S(z^{-1}, \bar{z}^{-1}) \\ g_N(z, \bar{z}) &= (z/\bar{z})\, g_S(z^{-1}, \bar{z}^{-1}) \end{aligned}$$

on $U_N \cap U_S$, and conclude that $\mathcal{A}^1(\mathbb{S}^2) \simeq \Gamma(\mathbb{S}^2, L^2 \oplus L^{-2})$.

Note that the last exercise now justifies the claim that the half-spin bundles were indeed $S^+ \oplus S^- \simeq L \oplus L^*$.

A.2.2 The spin connection ∇^S over $\mathbb{S}^2$

Given any local orthonormal basis of 1-forms $\{E_1, \dots, E_n\}$, we can compute Christoffel symbols with all three indices taken from this basis, by setting $\widehat{\Gamma}^\beta_{\mu\alpha} := (E_\mu)^i \widetilde{\Gamma}^\beta_{i\alpha}$, or equivalently, by requiring that

$$\nabla_{E_\mu} E_\alpha =: \widehat{\Gamma}^\beta_{\mu\alpha}\, E_\beta$$

for $\mu, \alpha, \beta = 1, 2, \dots, n$. (This works because the first index is tensorial).

Exercise A.13. *On U_N, take $z =: x^1 + ix^2$. Compute the ordinary Christoffel symbols Γ^k_{ij} in the (x^1, x^2) coordinates for the round metric $g = (4/q^2)(dx^1 \otimes dx^1 + dx^2 \otimes dx^2)$, and then show that*

$$\widehat{\Gamma}^{\beta}_{\mu\alpha} = \delta_{\mu\alpha} x^{\beta} - \delta_{\mu\beta} x^{\alpha} \quad \textit{for } \mu, \alpha, \beta = 1, 2.$$

This yields the local orthonormal bases $E_1 := \frac{1}{2}q\,\partial/\partial x^1$, $E_2 := \frac{1}{2}q\,\partial/\partial x^2$ for vector fields, and dually $\theta^1 = (2/q)\,dx^1$, $\theta^2 = (2/q)\,dx^2$ for 1-forms. However, since $\mathbb{S}^2 = \mathbb{C}P^1$ is a complex manifold, it is convenient to pass to "isotropic" bases, as follows. We introduce

$$E_+ := E_1 - iE_2 = q\frac{\partial}{\partial z}, \qquad \theta^+ := \tfrac{1}{2}(\theta^1 + i\theta^2) = \frac{dz}{q},$$
$$E_- := E_1 + iE_2 = q\frac{\partial}{\partial \bar{z}}, \qquad \theta^- := \tfrac{1}{2}(\theta^1 - i\theta^2) = \frac{d\bar{z}}{q}.$$

Exercise A.14. *Verify that the Levi-Civita connection on $\mathcal{A}^1(\mathbb{S}^2)$ is given, in these isotropic local bases, by*

$$\nabla_{E_+}\left(\frac{dz}{q}\right) = \bar{z}\,\frac{dz}{q}, \qquad \nabla_{E_-}\left(\frac{dz}{q}\right) = -z\,\frac{dz}{q},$$
$$\nabla_{E_+}\left(\frac{d\bar{z}}{q}\right) = -\bar{z}\,\frac{d\bar{z}}{q}, \qquad \nabla_{E_-}\left(\frac{d\bar{z}}{q}\right) = z\,\frac{d\bar{z}}{q}.$$

The Clifford action on spinors is given (over U_N, say) by $\gamma^1 := \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\gamma^2 := \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. The $\mathbb{Z}_2$-grading operator is given by

$$\chi := (-i)\,\sigma^1\sigma^2 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The spin connection is now specified by

$$\nabla^S_{E_\pm} := E_\pm - \tfrac{1}{4}\widehat{\Gamma}^{\beta}_{\pm\alpha}\,\gamma^{\alpha}\gamma_{\beta}.$$

Exercise A.15. *Verify that, over U_N, ∇^S is determined by*

$$\nabla^S_{E_+} = q\frac{\partial}{\partial z} + \tfrac{1}{2}\bar{z}\,\chi, \qquad \nabla^S_{E_-} = q\frac{\partial}{\partial \bar{z}} - \tfrac{1}{2}z\,\chi.$$

Conclude that the Dirac operator $D\!\!\!/ = -i\sigma^1\,\nabla^S_{E_1} - i\sigma^2\,\nabla^S_{E_2}$ is given, over U_N, by

$$D\!\!\!/ = -i\begin{pmatrix} 0 & q\frac{\partial}{\partial z} - \frac{1}{2}\bar{z} \\ q\frac{\partial}{\partial \bar{z}} - \frac{1}{2}z & 0 \end{pmatrix}.$$

A similar expression is valid over U_S, by replacing $z, \bar{z}, q$ by $\zeta, \bar{\zeta}, q'$ respectively, and by changing the overall $(-i)$ factor to $(+i)$. This formal change of sign is brought about by the local coordinate transformation formulas induced by $\zeta = 1/z$. (Here is an instance of the "unique continuation property" of $D\!\!\!/$: the local expression for the Dirac operator on any one chart determines its expressions on any overlapping chart, and then by induction, on the whole manifold.)

Exercise A.16. *By integrating spinor pairings with the volume form* $\nu = \sin\theta d\theta \wedge d\phi = 2iq^{-2}\, dz \wedge d\bar{z}$*, check that* $D\!\!\!/$ *is indeed symmetric as an operator on* $L^2(\mathbb{S}^2, S)$ *with domain* $\mathcal{S}$*.*

Exercise A.17. *Show that the spinor Laplacian* Δ^S *is given in the isotropic basis by*

$$\Delta^s = -\tfrac{1}{2}\big(\nabla^S_{E_+}\nabla^S_{E_-} + \nabla^S_{E_-}\nabla^S_{E_+} - z\nabla^S_{E_+} - \bar{z}\nabla^S_{E_-}\big),$$

and compute directly *that* $D\!\!\!/^2 = \Delta^S + \frac{1}{2}$*. This is consistent with the value* $s \equiv 2$ *of the scalar curvature of* $\mathbb{S}^2$*, taking into account how the metric* g *is normalized.*

A.2.3 Spinor harmonics and the Dirac operator spectrum

Newman and Penrose (1966) introduced a family of special functions on $\mathbb{S}^2$ that yield an orthonormal basis of spinors, in the same way that the conventional spherical harmonics Y_{lm} yield an orthonormal basis of L^2-functions. For functions, l and m are integers, but the spinors are labelled by "half-odd-integers" in $\mathbb{Z}+\frac{1}{2}$. When expressed in our coordinates $(z, \bar{z})$, they are given as follows.

For $l \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\} = \mathbb{N} + \frac{1}{2}$, and $m \in \{-l, -l+1, \dots, l-1, l\}$, write

$$Y^+_{lm}(z,\bar{z}) := C_{lm}\, q^{-l} \sum_{r-s=m-\frac{1}{2}} \binom{l-\frac{1}{2}}{r}\binom{l+\frac{1}{2}}{s} z^r(-\bar{z})^s,$$

$$Y^-_{lm}(z,\bar{z}) := C_{lm}\, q^{-l} \sum_{r-s=m+\frac{1}{2}} \binom{l+\frac{1}{2}}{r}\binom{l-\frac{1}{2}}{s} z^r(-\bar{z})^s,$$

where r, s are integers with $0 \le r \le l \mp \frac{1}{2}$ and $0 \le s \le l \pm \frac{1}{2}$ respectively; and

$$C_{lm} = (-1)^{l-m}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!(l-m)!}{(l+\frac{1}{2})!(l-\frac{1}{2})!}}.$$

Exercise A.18. *Show that* $Y^\pm_{lm}$ *are half-spinors in* $S^\pm$*, by applying the transformation laws under* $z \mapsto z^{-1}$ *and checking the regularity at the poles.*

Then define pairs of full spinors by

$$Y'_{lm} := \frac{1}{\sqrt{2}}\begin{pmatrix} Y^+_{lm} \\ iY^-_{lm}\end{pmatrix}, \qquad Y''_{lm} := \frac{1}{\sqrt{2}}\begin{pmatrix} -Y^+_{lm} \\ iY^-_{lm}\end{pmatrix}.$$

These turn out to be eigenspinors for the Dirac operator.

Exercise A.19. *Verify the following eigenvalue relations:*

$$D\!\!\!/ Y'_{lm} = (l+\tfrac{1}{2})\, Y'_{lm}, \qquad D\!\!\!/ Y''_{lm} = -(l+\tfrac{1}{2})\, Y''_{lm}.$$

Goldberg *et al* (1967) showed that these half-spinors are special cases of matrix elements $\mathcal{D}^l_{nm}$ of the irreducible group representations for $SU(2)$, namely,

$$Y^\pm_{lm}(z,\bar{z}) = \sqrt{\frac{2l+1}{4\pi}}\mathcal{D}^l_{\mp\frac{1}{2},m}(-\phi,\theta,-\phi),$$

By setting $h^\pm_{lm}(\theta,\phi,\psi) := e^{\mp\frac{1}{2}(\phi+\psi)} Y^\pm_{lm}(z,\bar{z})$, we get an orthonormal set of elements of $L^2(\mathrm{SU}(2))$, such that $\int_{\mathrm{SU}(2)} |h^\pm_{lm}(g)|^2\, dg = (1/4\pi)\int_{\mathbb{S}^2} |Y^\pm_{lm}|^2\, \nu$. The Plancherel formula for

SU(2) can then be used to show that these are a complete set of eigenvalues for $\not{D}$. Thus we have obtained the spectrum:

$$\mathrm{sp}(\not{D}) = \{\, \pm(l + \tfrac{1}{2}) : l \in \mathbb{N} + \tfrac{1}{2} \,\} = \{\pm 1, \pm 2, \pm 3, \dots\} = \mathbb{N} \setminus \{0\},$$

with respectively multiplicities $(2l+1)$ in each case, since the index m in $Y^{\pm}_{lm}$ takes $(2l+1)$ distinct values.

Postscript: Since $s \equiv 2$ and $\not{D}^2 = \Delta^S + \frac{1}{2}$, we also get

$$\mathrm{sp}(\Delta^S) = \{\, (l + \tfrac{1}{2})^2 - \tfrac{1}{2} = l^2 + l - \tfrac{1}{4} : l \in \mathbb{N} + \tfrac{1}{2} \,\}$$

with multiplicities $2(2l+1)$ in each case. Note that

$$\mathrm{sp}(\not{D}^2) = \{\, (l + \tfrac{1}{2})^2 = l^2 + l + \tfrac{1}{4} : l \in \mathbb{N} + \tfrac{1}{2} \,\}.$$

The operator $\underline{C}$ given by $\underline{C} := \Delta^S + \frac{1}{4} = \not{D} - \frac{1}{4}$ has spectrum

$$\mathrm{sp}(\underline{C}) = \{\, l(l+1) : l \in \mathbb{N} + \tfrac{1}{2} \,\},$$

with multiplicities $2(2l+1)$ again. This $\underline{C}$ comes from the *Casimir element* in the centre of $\mathcal{U}(\mathfrak{su}(2))$, represented on $\mathcal{H} = L^2(\mathbb{S}^2, S)$ via the rotation action of SU(2) on the sphere $\mathbb{S}^2$. There is a general result for compact symmetric spaces $M = G/K$ with a G-invariant spin structure, namely that $\not{D} = \underline{C}_G + \frac{1}{8}s$, or $\Delta^S = \underline{C}_G - \frac{1}{8}s$. This is a nice companion result, albeit only for homogeneous spaces, to the Schrödinger–Lichnerowicz formula. Details are given in Section 3.5 of Friedrich's book.

A.3 Spin$^{\mathrm{c}}$ Dirac operators on the 2-sphere

We know that finitely generated projective modules over the C^*-algebra $A = C(\mathbb{S}^2)$ are of the form $p\,A^k$, where $p = [p_{ij}]$ is an $k \times k$ matrix with elements in A, such that $p\ (= p^2 = p^*)$ is an orthogonal projector, whose rank is $\operatorname{tr} p = p_{11} + \cdots + p_{kk}$. To get modules of sections of *line bundles*, we impose the condition that $\operatorname{tr} p = 1$, so that $p\,A^k$ is an A-module "of rank one". It turns out that it is enough to consider the case $k = 2$ of 2×2 matrices.

Exercise A.20. *Check that any projector $p \in M_2(C(\mathbb{S}^2))$ is of the form*

$$p = \frac{1}{2}\begin{pmatrix} 1 + n_3 & n_1 - in_2 \\ n_1 + in_2 & 1 - n_3 \end{pmatrix},$$

where $n_1^2 + n_2^2 + n_3^2 = 1$, so that $\vec{n} = (n_1, n_2, n_3)$ is a continuous function from $\mathbb{S}^2$ to $\mathbb{S}^2$.

After stereographic projection, we can replace $\vec{n}$ by $f(z) := \dfrac{n_1 - in_2}{1 - n_3}$. where $z = e^{-i\phi}\cot\frac{\theta}{2}$ is allowed to take the value $z = \infty$ at the north pole. Then f is a continuous map from the Riemann sphere $\mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$ into itself. If two projectors p and q are homotopic —there is a continuous path of projectors $\{\, p_t : 0 \leq t \leq 1 \,\}$ with $p_0 = p$ and $p_1 = q$— then they give the same class $[p] = [q]$ in $K^0(\mathbb{S}^2)$; and this happens if and only if the corresponding maps $\vec{n}$, or functions $f(z)$, are homotopic.

Exercise A.21. *Consider, for each $m = 1, 2, 3, \dots$, the maps*

$$z \mapsto f_m(z) := z^m \quad \textit{and} \quad z \mapsto f_{-m}(z) := \bar{z}^m$$

of the Riemann sphere into itself. ¿Can you describe the corresponding maps $\vec{n}$ of $\mathbb{S}^2$ into itself? Can you show that any two of these maps are not homotopic?

Let $\mathcal{E}_{(m)} = p_m\, A^2$ and $\mathcal{E}_{(-m)} = p_{-m}\, A^2$, where

$$p_m(z) = \frac{1}{1 + z^m \bar{z}^m} \begin{pmatrix} z^m \bar{z}^m & z^m \\ \bar{z}^m & 1 \end{pmatrix}, \qquad p_{-m}(z) = \frac{1}{1 + z^m \bar{z}^m} \begin{pmatrix} z^m \bar{z}^m & \bar{z}^m \\ z^m & 1 \end{pmatrix},$$

with the obvious definition (¿what is it?) for $z = \infty$.

Exercise A.22. *Show that $\mathcal{E}_{(1)}$ is isomorphic to the space of sections of the tautological line bundle $L \to \mathbb{C}P^1$ [hint: apply p_1 to any element of $\mathcal{A}^2$ and examine the result]. Show also that $\mathcal{E}_{(-1)}$ gives the space of sections of the dual line bundle $L^* \to \mathbb{C}P^1$.*

Exercise A.23. *For $m = 2, 3, \dots$, show that $\mathcal{E}_{(m)} \simeq \mathcal{E}_{(1)} \otimes_A \cdots \otimes_A \mathcal{E}_{(1)}$ (m times) by examining the components of elements of $p_m A^2$. ¿What is the analogous result for $\mathcal{E}_{(-m)}$?*

For $m \in \mathbb{Z}$, $m \neq 0$, we redefine $\mathcal{E}_{(m)} := p_m \mathcal{A}^2$ with $\mathcal{A} = C^\infty(\mathbb{S}^2)$; so that $\mathcal{E}_{(m)}$ now denotes *smooth* sections over a nontrivial line bundle on $\mathbb{S}^2$. We can identify each element of $\mathcal{E}_{(m)}$ with a smooth function $f_N \colon U_N \to \mathbb{C}$ for which there is another smooth function $f_S \colon U_S \to \mathbb{C}$, such that

$$f_N(z) = (\bar{z}/z)^{m/2}\, f_S(z^{-1}) \quad \text{for all } z \neq 0. \qquad (\underline{m})$$

Here, as before, $(\bar{z}/z)$ means $e^{i\phi}$ in polar coordinates.

Exercise A.24. *Writing $E_+ := q\,\partial/\partial z$ and $E_- := q\,\partial/\partial \bar{z}$ as before, where $q = 1 + z\bar{z}$, show that when the operators*

$$\nabla^{(m)}_{E_+} = q\,\frac{\partial}{\partial z} + \tfrac{1}{2} m \bar{z}, \qquad \nabla^{(m)}_{E_-} = q\,\frac{\partial}{\partial \bar{z}} - \tfrac{1}{2} m z,$$

are applied to functions f_N that satisfy $(\underline{m})$, the image also satisfies $(\underline{m})$. Thus they are components of a connection $\nabla^{(m)}$ on $\mathcal{E}_{(m)}$.

To get all the spin$^{\mathrm{c}}$ structures on $\mathbb{S}^2$, we twist the spinor module $\mathcal{S}$ for the spin structure, namely $\mathcal{S} = \mathcal{E}_{(1)} \oplus \mathcal{E}_{(-1)}$, by the rank-one module $\mathcal{E}_{(m)}$. On the tensor product $\mathcal{S} \otimes_{\mathcal{A}} \mathcal{E}_{(m)}$ we use the connection

$$\nabla^{S,m} := \nabla^S \otimes 1_{\mathcal{E}_{(m)}} + 1_{\mathcal{S}} \otimes \nabla^{(m)}.$$

Exercise A.25. *Show that the Dirac operator $D\!\!\!/_m := -i\,\hat{c} \circ \nabla^{S,m}$, that acts on $\mathcal{S} \otimes_{\mathcal{A}} \mathcal{E}_{(m)}$, is given by*

$$D\!\!\!/_m \equiv \begin{pmatrix} 0 & D\!\!\!/_m^- \\ D\!\!\!/_m^+ & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & q\frac{\partial}{\partial z} + \frac{1}{2}(m-1)\,\bar{z} \\ q\frac{\partial}{\partial \bar{z}} - \frac{1}{2}(m+1)\,z & 0 \end{pmatrix}.$$

Check also that

$$D\!\!\!/_m^+ = -i\, q^{(m+3)/2}\, \frac{\partial}{\partial \bar{z}}\, q^{-(m+1)/2} \quad \textit{and} \quad D\!\!\!/_m^- = -i\, q^{-(m-3)/2}\, \frac{\partial}{\partial z}\, q^{(m-1)/2},$$

where these powers of q are multiplication operators on suitable spaces on functions on U_N.

Exercise A.26. *If $m < 0$, show that any element of* $\ker \not{D}_m^+$ *is of the form* $a(z)\, q^{(m+1)/2}$ *where* $a(z)$ *is a holomorphic polynomial of degree* $< |m|$. *Also, if* $m \geq 0$, *show that* $\ker \not{D}_m^+ = 0$.

Exercise A.27. *If $m > 0$, show that any element of* $\ker \not{D}_m^-$ *is of the form* $b(\bar{z})\, q^{-(m-1)/2}$ *where* $b(\bar{z})$ *is an* anti*holomorphic polynomial of degree* $< m$. *Also, if* $m \leq 0$, *show that* $\ker \not{D}_m^- = 0$. *Conclude that the index of* $\not{D}_m$ *equals* $-m$ *in all cases.*

The *sign* of a selfadjoint operator D on a Hilbert space is given by the relation $D =: F\,|D| = F\,(D^2)^{1/2}$, where we put $F := 0$ on $\ker D$. Thus F is a bounded selfadjoint operator such that $1 - F^2$ is the orthogonal projector whose range is $\ker D$. When $\ker D$ is finite-dimensional, $1 - F^2$ has finite rank, so it is a compact operator.

An *even Fredholm module* over an algebra $\mathcal{A}$ is given by:

1. a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$;
2. a representation $a \mapsto \pi(a) = \begin{pmatrix} \pi^0(a) & 0 \\ 0 & \pi^1(a) \end{pmatrix}$ of $\mathcal{A}$ on $\mathcal{H}$ by bounded operators that commute with the $\mathbb{Z}_2$-grading;
3. a selfadjoint operator $F = \begin{pmatrix} 0 & F^- \\ F^+ & 0 \end{pmatrix}$ on $\mathcal{H}$ that anticommutes with the $\mathbb{Z}_2$-grading, such that $F^2 - 1$ and $[F, \pi(a)]$ are compact operators on $\mathcal{H}$, for each $a \in \mathcal{A}$.

We can extend the twisted Dirac operator $\not{D}_m$ to a selfadjoint operator on $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$, where $\mathcal{H}^0$ and $\mathcal{H}^1$ are two copies of the Hilbert space $L^2(\mathbb{S}^2, \nu)$ where $\nu = 2i\, q^{-2}\, dz\, d\bar{z}$. We define $\pi^0(a) = \pi^1(a)$ to be the usual multiplication operator of a function $a \in C^\infty(\mathbb{S}^2)$ on this L^2-space.

Exercise A.28. *Show that* $\not{D}_m$, *given by the above formulas on its original domain, is a symmetric operator on* $\mathcal{H}$.

Exercise A.29. *Check that the sign* F_m *of the twisted Dirac operator* $\not{D}_m$ *determines a Fredholm module over* $C^\infty(\mathbb{S}^2)$.