第五章習作

1. **Spheres:**

   (a) Compute the volume of the sphere $S^n_r := \{x \in \mathbb{R}^{n+1} : |x| = r\}$ of radius r.

   (b) Compute the second fundamental form of the sphere $S^n \subset \mathbb{R}^{n+1}$.

   (c) Here is an exercise that requires a more elaborate computation: Consider the circle $S^1_{1/\sqrt{2}} := \{(y^1, y^2) \in \mathbb{R}^2 : (y^1)^2 + (y^2)^2 = \frac{1}{2}\}$ (why the subscript $\sqrt{2}$?) and the so-called Clifford torus $C := S^1_{1/\sqrt{2}} \times S^1_{1/\sqrt{2}} \subset \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$. Show that $C \subset S^3$ is a minimal submanifold.

   (d) Show that the equator $S^{n-1} = \{(y^1, \dots, y^n) \in \mathbb{R}^n : \sum_{i=1}^n (y^i)^2 = 1\} \subset S^n = \{(x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} (x^i)^2 = 1\}$ is totally geodesic.

2. **Hyperbolic spaces:** Similarly, find a totally geodesic submanifold of hyperbolic space H^n.

3. **Tori:** Consider subtori of a torus as in the exercises for Chap. 1. Show that they are totally geodesic.

1. Consider the hyperboloid in $\mathbb{R}^3$ defined by the equation

$$x^2 + y^2 - z^2 = -1, z > 0$$

   and compute its curvature.

2. Verify that the catenoid, the helicoid, and Enneper's surface are minimal surfaces.
3. Determine all surfaces of revolution in $\mathbb{R}^3$ that are minimal. (Answer: The catenoid is the only one.)
4. Let $F : M^m \to \mathbb{R}^{m+1}$ be an isometric immersion ($m = \dim M$). Give a complete derivation of the formula

$$\Delta F = m\eta$$

   where Δ is the Laplace–Beltrami operator of M and η is the mean curvature vector of $F(M)$.
5. Let $F : M^m \to S^n \subset \mathbb{R}^{n+1}$ be an isometric immersion. Show that $F(M)$ is minimal in S^n if and only if there exists a function φ on M with $\Delta F = \varphi F$ and that in this case necessarily $\varphi \equiv m$.
6. Show that for $n \geq 4$, there exists no hypersurface (i.e. a submanifold of codimension 1) in $\mathbb{R}^n$ with negative sectional curvature.