第三章 習作

1. S^n ： R^{n+1} 上的極座標 $(r,\phi^1,...,\phi^n)$

(a)The Laplace operator Δ on S^n on functions $f:S^n \to R$ can be obtained from Euclidean Laplace operator Δ_0 on R^{n+1} via

$$\Delta f = \Delta_0 f + n\frac{\partial F}{\partial r} + \frac{\partial^2 F}{\partial r^2}$$

極座標為 $(r,\phi^1,...,\phi^n)$ ，F是f在 R^{n+1} 的平滑延拓，Laplace為 Δ_0 則

$$\Delta_0 = \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^n}$$ （通常取 $F(r,\phi^1,...,\phi^n) = f(\phi^1,...,\phi^n)$）

例 n=2 $\Delta_{S^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$

取 $f(\theta,\phi) = \cos\theta$ 則 $\Delta_{S^2}\cos\theta = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta(-\sin\theta)) = ... = -2\cos\theta$

$$\Delta_0 F = \frac{1}{r^2}(-2\cos\theta) = -\frac{2\cos\theta}{r^2}$$

註：

極座標 (r,θ) $\Delta = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial^2\theta}$

球座標 (r,θ,ϕ) $\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$

(b)Let p be a homogeneous polynomial on R^{n+1} of degree k that is harmonic，i.e. $\Delta_0 p = 0$。Then its restriction to S^n satisfies $\Delta p = k(k+n-1)p$。In particular，such a p then is an eigenfunction of S^n for the eigenvalue k(k+n-1)。
(That is the resaen why such polynomials are called spherical，the spectrum of S^n consists of the values k(k+n-1) for k=0,1,⋯)

[Spec403-1]球諧函數 $\Delta_{S^n} p = 0$ $p(tx) = t^k p(x)$ （ $p(r,\theta) = r^k f_k(\theta)$ ）

[Spec403-2]

(c)Compute the Laplace operator of S^n on p-forms ($0 \le p \le n$)in the stereograpgic coordinates $(x^1,...,x^n)$

在 S^n 上Laplace de Rham operator的定義為 $\Delta = d\delta + \delta d$

使用立體投影座標計算 p=2 的情形

(d)Let $\omega \in \Omega^1(S^2)$ be a 1-form on S^2。

Suppose $\varphi^*\omega = \omega$ for all $\varphi \in SO(3)$。Show that $\omega \equiv 0$

Formulate and prove a general result for invariant differential forms on S^n。

2. **Tori:** A *torus* $T^n := \pi(\mathbb{R}^n)$ was obtained by taking linearly independent vectors $w_1, w_2, \ldots, w_n \in \mathbb{R}^n$ and considering $z_1, z_2 \in \mathbb{R}^n$ as equivalent if there are $m_1, m_2, \ldots, m_n \in \mathbb{Z}$ with

$$z_1 - z_2 = \sum_{i=1}^{n} m_i w_i.$$

The points of the form $w = \sum_{i=1}^{n} m_i w_i$ constitute a subgroup of the group of translations of $\mathbb{R}^n$. Such a subgroup is called a *lattice*. Thus, a torus T^n can be seen as a quotient $\mathbb{R}^n/\Gamma$ for some such lattice Γ.
The Laplacian on a torus T^n is directly obtained from that of $\mathbb{R}^n$.

2. Tori：

(a)The eigenvalues of the Laplacian on a circle of length L are

$\lambda_k = \dfrac{4\pi^2 k^2}{L^2}, k = 0,1,2,...$ each with multiplicity 2。

The eigenfunctions are obtained from the trigonometric

functions $\sin(\dfrac{Lx}{2\pi}), \cos(\dfrac{Lx}{2\pi})$ on R。

This follows from elementary Fourier analysis。

(b)

(b) Consider a torus $T^n = \mathbb{R}^n/\Gamma$. Denote the elements of Γ by w. The dual lattice Γ^* then consists of those vectors $w^* \in \mathbb{R}^n$ that satisfy

$$\langle w, w^* \rangle \in \mathbb{Z} \quad \text{for all } w \in \Gamma, \tag{3.5.25}$$

where $\langle .,. \rangle$ denotes the Euclidean scalar product. The functions

$$f_{w^*}(x) := \exp(2i\langle w^*, x\rangle) \text{ for } w^* \in \Gamma^* \tag{3.5.26}$$

then are eigenfunctions on T^n with eigenvalues $4\pi^2|w^*|^2$. (They are eigenfunctions on $\mathbb{R}^n$ that are invariant under Γ and therefore project to functions on T^n. As in the case of S^1, one shows with the help of elementary Fourier analysis that these are all the eigenfunctions of T^n.)

1. Give a detailed proof of the formula

$$*\Delta = \Delta * .$$

2. Let M be a two-dimensional Riemannian manifold. Let the metric be given by $g_{ij}(x)dx^i \otimes dx^j$ in local coordinates (x^1, x^2). Compute the Laplace operator on 1-forms in these coordinates. Discuss the case where

$$g_{ij}(x) = \lambda^2(x)\delta_{ij}$$

with a positive function $\lambda^2(x)$.

3. Suppose that $\alpha \in H_p^{1,2}(M)$ satisfies

$$(d^*\alpha, d^*\varphi) + (d\alpha, d\varphi) = (\eta, \varphi) \quad \text{for all} \quad \varphi \in \Omega^p(M),$$

with some given $\eta \in \Omega^p(M)$. Show $\alpha \in \Omega^p(M)$, i.e. smoothness of α.

4. The considerations of the spectrum of the Laplace–Beltrami operator on functions as given in Sect. 3.2 can be extended to differential forms, as briefly described and utilized in Sect. 3.3. The present exercise leads you to the systematic derivation of these results.

   Thus, let M be a compact oriented Riemannian manifold, and let Δ be the Laplace operator on $\Omega^p(M)$. $\lambda \in \mathbb{R}$ is called an eigenvalue if there exists some $u \in \Omega^p(M), u \neq 0$, with

$$\Delta u = \lambda u.$$

   Such a u is called an eigenform or an eigenvector corresponding to λ. The vector space spanned by the eigenforms for λ is denoted by V_λ and called the eigenspace for λ.

   Show:

   (a) All eigenvalues of Δ are nonnegative.
   (b) All eigenspaces are finite dimensional.
   (c) The eigenvalues have no finite accumulation point.
   (d) Eigenvectors for different eigenvalues are orthogonal.

   As in Sect. 3.2, the next results need a little more analysis (cf. e.g. [242]).

   (e) There exist infinitely many eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$$

   (f) All eigenvectors of Δ are smooth.
   (g) The eigenvectors of Δ constitute an L^2-orthonormal basis for the space of p-forms of class L^2.

5. Here is another long exercise:

   Let M be a compact oriented Riemannian manifold with boundary $\partial M \neq \emptyset$. For $x \in \partial M$, $V \in T_xM$ is called tangential if it is contained in $T_x\partial M \subset T_xM$ and $W \in T_xM$ is called normal if

$$\langle V, W \rangle = 0 \quad \text{for all tangential} \quad V.$$

   An arbitrary $Z \in T_xM$ can then be decomposed into a tangential and a normal component:

$$Z = Z_{\tan} + Z_{\text{nor}}.$$

Analogously, $\eta \in \Gamma^p(T_x^* M)$ can be decomposed into

$$\eta = \eta_{\mathrm{tan}} + \eta_{\mathrm{nor}}$$

where η_{tan} operates on tangential p-vectors and η_{nor} on normal ones. For p-forms ω on M, we may impose the so-called absolute boundary conditions

$$\left.\begin{aligned} \omega_{\mathrm{tan}} &= 0, \\ (d^*\omega)_{\mathrm{nor}} &= 0, \end{aligned}\right. \qquad \text{on } \partial M,$$

or the relative boundary conditions

$$\left.\begin{aligned} \omega_{\mathrm{nor}} &= 0, \\ (d\omega)_{\mathrm{nor}} &= 0, \end{aligned}\right. \qquad \text{on } \partial M.$$

(These two boundary conditions are interchanged by the $*$-operator.)

Develop a Hodge theory under either set of boundary conditions.