§ Geodesics and Jacobi fields

6.1 弧長與能量的第一、第二變分

$$L(s)=\int_a^b \left\langle \frac{\partial c(t,s)}{\partial t},\frac{\partial c(t,s)}{\partial t}\right\rangle^{\frac{1}{2}} dt \quad E(s)=\frac{1}{2}\int_a^b \left\langle \frac{\partial c(t,s)}{\partial t},\frac{\partial c(t,s)}{\partial t}\right\rangle dt$$

Theorem 6.1.1 *Let $c:[a,b]\to M$ be geodesic. Then*

$$E''(0)=\int_a^b \langle \nabla_{\frac{\partial}{\partial t}} c'(t,0),\nabla_{\frac{\partial}{\partial t}} c'(t,0)\rangle dt-\int_a^b \langle R(\dot c,c')c',\dot c\rangle dt_{|s=0}+\langle \nabla_{\frac{\partial}{\partial s}} c',\dot c\rangle|_{t=a,s=0}^{t=b,s=0} \tag{6.1.6}$$

and with $c'^{\perp}:=c'-\langle \frac{\dot c}{\|\dot c\|},c'\rangle \frac{\dot c}{\|\dot c\|}$ (the component of c' orthogonal to $\dot c$),

$$L''(0)=\frac{1}{\|\dot c\|}\left\{\int_a^b (\langle \nabla_{\frac{\partial}{\partial t}} c'^{\perp},\nabla_{\frac{\partial}{\partial t}} c'^{\perp}\rangle-\langle R(\dot c,c'^{\perp})c'^{\perp},\dot c\rangle)\,dt+\langle \nabla_{\frac{\partial}{\partial s}} c',\dot c\rangle|_{t=a}^{t=b}\right\}\Big|_{s=0}. \tag{6.1.7}$$

定理 6.1.2 Synge

Any compact oriented even-dimensional Riemannian manifold with positive sectional curvature is simply connected。

§ 定義 Index form of the geodesic

$I(X,Y):=\int_a^b (\langle \nabla_T X,\nabla_T Y\rangle-\langle R(\dot c,X)Y,\dot c\rangle)dt$ where $T=\frac{\partial}{\partial t}$

即 $I(X,X)=\frac{d^2}{ds^2}E(0)$ if X(a)=0=X(b)

定理 6.2.1

Let $c:[0,T]\to M$ be a geodesic。

Let c(t,s) be a variation of c(t)

$c(t,s):[0,T]\times(-\varepsilon,\varepsilon)\to M$ ，for which all curves c(t,s) are geodesics too。Then $X(t):=\frac{\partial}{\partial s}c(t,s)|_{s=0}$ is a Jacobi field along c(t)。

推論

Every Killing field K on M is a Jacobi field along any geodesic c in M。

6.2.2 推論 假設$\dot{c}(0)=v$

Let $c:[0,T]\to M$ be a geodesic，p=c(0)，i.e. $c(t)=\exp_p(tv)$

For $w\in T_pM$，the Jacobi field X along c with X(0)=0，$\dot{X}(0)=w$ then is given by $X(t)=(D\exp_p)(tv))(tw)$ (或者寫成 $D_{tv}\exp_p(tw)$)

(The derivative of the exponential map $\exp_p:T_pM\to M$，evaluated at the point $tv\in T_pM$ and applied to tw。)

(1) 在切空間T_pM中取一個向量 w。考慮一條方向向量的曲線 v(s)=v+tw

(2) 定義一族測地線$c(s,t)=\exp_p(t\cdot v(s))=\exp_p(t(v+sw))$

(3) Jacobi 場 X(t)的定義

$$X(t)=\frac{\partial}{\partial s}\Big|_{s=0}\exp_p(t(v+sw))$$

(4) 根據多元微積分的連鎖律，上述導數即為指數映射$\exp_p$在點 tv 處的導數

$(D\exp_p)_{tv}$，作用於切空間中的方向向量 tw。$X(t)=(D\exp_p)_{tv}(tw)$

解釋連鎖律： $\gamma(s)=t(v+sw)$

外函數 $\exp_p:T_pM\to M$ 內函數 $\gamma(s):R\to T_pM$

$$X(t)=(D\exp)_{\gamma(0)}(\gamma'(0))$$

$\gamma(0)=t(v+0\cdot w)=tv$ ，$\gamma'(s)=\dfrac{d}{ds}(tv+tsw)=tw$，$\gamma'(0)=tw$

(Dexp_p)是指數映射的導數，將「切空間（T_pM）裡的向量」轉換為「流形 M 上的切向量」。

對於半徑為 1 的單位球面S^2，其截面曲率恆為 1。在這種情況下，曲率算子有一個非常簡單的形式：

$R(X,\dot{c})\dot{c}=\left\langle \dot{c},\dot{c}\right\rangle X-\left\langle X,\dot{c}\right\rangle \dot{c}$ 選取的 w 與測地線方向 $\dot{c}$ 垂直，且測地線速度為單位長度)，上述算子簡化為： $R(X,\dot{c})\dot{c}=X(t)$ 代入 Jacobi 方程得到

$\ddot{X}(t)+X(t)=0$ 且 X(0)=0， $\dot{X}(0)=w$ 所以 $X(t)=(\sin t)w$

現在我們把這個解與文件中的公式對接： $X(t)=(D\exp_p)_{tv}(tw)$

$(D\exp_p)$ 產生 $\dfrac{\sin t}{t}$ 縮放： $(D\exp_p)$ (長度 t 的向量)→得到長度 sin(t)的向量。

推論 Gauss lemma

Corollary 6.2.3 *Let $p\in M, v\in T_pM, c(t):=\exp_p tv$ the geodesic with $c(0)=p, \dot{c}(0)=v$ $(t\in[0,1])$, assuming that v is contained in the domain of definition of $\exp_p$. Then for any $w\in T_pM$*

$$\langle v,w\rangle=\langle (D_v\exp_p)v,(D_v\exp_p)w\rangle, \qquad (6.2.10)$$

where $D_v\exp_p$, the derivative of $\exp_p$ at the point v, is applied to the vectors v and w considered as vectors tangent to T_pM at the point v.